

QUANTUM DISSIPATIVE CHAOS

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ABSTRACT

Using the decoherence formalism of Gell-Mann and Hartle, a quantum system is found which is the equivalent of the classical dissipative chaotic Duffing oscillator. The similarities and differences from the classical oscillator are examined; in particular, a new concept of quantum maps is introduced, and alterations in the classical strange attractor due to the presence of scale-dependent quantum effects are studied. Classical quantities such as the Lyapunov exponents and fractal dimension are examined, and quantum analogs are suggested. These results are generalized into a framework for quantum dissipative chaos, and there is a brief discussion of other work in this area.

I. INTRODUCTION

Since classical chaos first began to be studied, a conspicuous puzzle has been how to reconcile this nonlinear, purely classical phenomenon with an underlying linear quantum theory. If we believe, as we must to be consistent, that all of physics is fundamentally quantum mechanical in nature, then we must further believe that true chaotic systems do not exist. At some point, at length scales determined by Planck's constant, the deterministic uncertainties of classical chaos must give way to the probabilistic uncertainties of quantum mechanics.

But tackling these problems is not simple. The nonlinear equations of chaos are, in general, only solvable with modern high-speed computers, and their quantum analogs share this limitation. Also, chaos itself encompasses two major types of behavior: Hamiltonian chaos in which energy is conserved, and dissipative chaos. It is in principle straightforward to find and solve quantum equivalents to Hamiltonian systems, if difficult in practice, and considerable progress has been made in recent years in understanding these systems. Dissipative systems are much more foreign to quantum mechanics as it is usually studied.

Recently, Murray Gell-Mann and James Hartle have used the decoherence functional formalism of quantum mechanics to show how quasiclassical laws can arise from an underlying quantum theory [1–3]. I applied this approach to the problem of Brownian motion, demonstrating how their scheme reproduces exactly the classical Langevin equation in a fairly broad class of systems [4].

A natural next step is to apply this to systems with interesting classical behavior. Since dissipation is easily and indeed naturally included in such systems, an obvious candidate for study is dissipative chaos. Once a quantum system is found whose limiting behavior is equivalent to a classical chaotic system, we can study how the residual quantum mechanical effects alter the system, and what difference this makes to the classical behavior.

In section II we briefly examine the family of quantum systems from which we will draw our model, and derive the quasiclassical equations of motion for them. We then go to the

limit of an infinite reservoir of oscillators with a continuum of frequencies, and specialize to the case of a forced, damped nonlinear oscillator.

In section III we examine the classical behavior of one such system, the damped, driven Duffing oscillator. There is a brief discussion of dissipative chaos, the structure of the strange attractor, and the bifurcations leading to chaos. Several quantities useful for characterizing the chaotic behavior are defined: the fractal dimension and Lyapunov exponents, and their relationships are examined.

In section IV we look at the decoherence functional, and define the idea of a quantum map. The system is examined as a Wigner distribution, and we see how the invariant measure of the strange attractor goes over to the quantum case. Problems of coarse-graining and decoherence are discussed. Then we look at the system from a Master equation point of view, and compare this description to the decoherence functional approach. In section V we see how the various classical quantities used to characterize chaotic behavior can be reinterpreted for our quantum system, by treating it as a classical system with noise for sufficiently coarse length scales.

A few other treatments of quantum dissipative chaos are mentioned in section VI, and the differences between Hamiltonian and dissipative chaos are pointed out. Finally, in section VII a case is made for a general theory of quantum dissipative chaos.

II. DAMPED DRIVEN QUANTUM SYSTEMS.

A. The Quantum Systems.

Picking a good set of candidate systems requires some thought. Many widely studied sets of chaotic equations have only a loose connection to actual physical systems; many others are extreme coarse grainings of very complicated systems with many degrees of freedom, e.g., fluid dynamics. It is much better to deal with comparatively simple systems, whose decoherence functionals can be calculated easily. For this reason, I have elected to study

damped driven nonlinear oscillators, which can be easily modelled as particles moving in a potential well, interacting with a reservoir of simple linear oscillators. In particular, I will concentrate on one such system, the damped, driven Duffing oscillator.

Earlier work has chiefly considered systems with reservoirs in a thermal state. For the purpose of this model, I wish to consider instead a system whose reservoir is initially in a coherent state.

Consider a system of N harmonic oscillators. We assume them to be in a state $|\{\nu\}\rangle$, where $\{\nu\}$ represents a set of N complex numbers ν_j . If \hat{a}_i is the annihilation operator for the i th oscillator, then $\hat{a}_i|\{\nu\}\rangle = \nu_i|\{\nu\}\rangle$.

As shown in earlier papers [3,4], the decoherence functional for a system interacting with a reservoir is

$$\begin{aligned}
D[x'(t), x(t)] &= \exp\left\{i(S_{\text{sys}}[x'(t)] - S_{\text{sys}}[x(t)])/ \hbar\right\} \int \delta Q' \delta Q \delta(Q'(t_f) - Q(t_f)) \\
&\quad \times \exp\left\{i(S_{\text{res}}[Q'(t)] - S_{\text{res}}[Q(t)] \right. \\
&\quad \left. - \int_{t_0}^{t_f} (V(x'(t), Q'(t)) - V(x(t), Q(t))) dt / \hbar\right\} \\
&\quad \times \rho(x'_0, Q'_0; x_0, Q_0) \\
&= \exp\left\{i(S_{\text{sys}}[x'(t)] - S_{\text{sys}}[x(t)] + W[x'(t), x(t)]) / \hbar\right\} \tilde{\rho}(x'_0; x_0). \tag{2.1}
\end{aligned}$$

Here $S_{\text{sys}}[x(t)]$ is the action of the system for a given trajectory $x(t)$, $S_{\text{res}}[Q(t)]$ is the action of the reservoir for a given trajectory $Q(t)$, and $V(x, Q)$ is the interaction potential between the system and reservoir variables. We will assume this to be a bilinear potential of the form

$$V(x, Q) = - \sum_k \gamma_k x Q^k, \tag{2.2}$$

where Q^k is the coordinate of the k th oscillator in the reservoir. We will shortly allow the number of oscillators to go to infinity, and assume a continuum of oscillator frequencies, but for now let us deal with the discrete case.

We will also assume that the density matrix factors:

$$\rho(x'_0, Q'_0; x_0, Q_0) = \chi(x'_0; x_0) \phi(Q'_0; Q_0), \tag{2.3}$$

where $\phi(Q'_0; Q_0) = \langle Q'_0 | \{\nu\} \rangle \langle \{\nu\} | Q_0 \rangle$ is the pure coherent state described above.

We can readily calculate the influence functional for this system. It is just

$$\begin{aligned} \exp\left\{iW[x'(t), x(t)]/\hbar\right\} &= \int \int \int dQ_f dQ_0 dQ'_0 K_{x'(t)}^*(Q_f; Q'_0) K_{x(t)}(Q_f; Q_0) \phi(Q'_0; Q_0) \\ &= \langle \{\nu\} | \hat{S}_{x'(t)}^\dagger \hat{S}_{x(t)} | \{\nu\} \rangle, \end{aligned} \quad (2.4)$$

where $K_{x(t)}(Q_f; Q_0)$ is the transition amplitude from Q_0 to Q_f of the reservoir and $\hat{S}_{x(t)}$ is the time evolution operator of a forced harmonic oscillator driven by the time-dependant interaction $V(x(t), Q)$ given in (2.2). This is a well-known problem [5]. For a single oscillator of frequency ω the operator is

$$\hat{S}_{x(t)} = \exp\left[\alpha \hat{a}^\dagger - \alpha^* \hat{a}\right] = D(\alpha), \quad (2.5)$$

where

$$\alpha = \frac{i\gamma}{\sqrt{2m\omega\hbar}} \int_{t_0}^{t_f} e^{i\omega s} x(s) ds, \quad (2.6)$$

and

$$D(\alpha)|\nu\rangle = |\nu + \alpha\rangle e^{i\text{Im}(\alpha\nu^*)}. \quad (2.7)$$

For $\nu = 0$ this just reduces to the usual form of the influence functional for an oscillator initially in the ground state:

$$\begin{aligned} \exp\left\{iW[x'(t), x(t)]/\hbar\right\} &= \exp\frac{i}{\hbar}\left\{+\frac{i\gamma^2}{4m\omega} \int_{t_0}^{t_f} dt \int_{t_0}^{t_f} ds \cos(\omega(t-s))(x'(t)-x(t))(x'(s)-x(s))\right. \\ &\quad \left.-\frac{\gamma^2}{2m\omega} \int_{t_0}^{t_f} dt \int_{t_0}^t ds \sin(\omega(t-s))(x'(t)-x(t))(x'(s)+x(s))\right\}. \end{aligned} \quad (2.8)$$

For non-zero ν we get an additional exponent of the form

$$\text{Im}(2\nu\alpha^* + 2\nu^*\alpha') = \gamma\sqrt{\frac{2}{m\omega\hbar}} \int_{t_0}^{t_f} dt (\text{Re}\nu \cos(\omega t) + \text{Im}\nu \sin(\omega t))(x(t) - x'(t)). \quad (2.9)$$

Generalizing this to many oscillators, we get the influence phase

$$W[x'(t), x(t)] = \sum_k \frac{i\gamma_k^2}{4m\omega_k} \int_{t_0}^{t_f} dt \int_{t_0}^{t_f} ds \cos(\omega_k(t-s))(x'(t)-x(t))(x'(s)-x(s))$$

$$\begin{aligned}
& -\frac{\gamma_k^2}{2m\omega_k} \int_{t_0}^{t_f} dt \int_{t_0}^t ds \sin(\omega_k(t-s))(x'(t) - x(t))(x'(s) + x(s)) \\
& + \gamma_k \sqrt{\frac{2\hbar}{m\omega_k}} \int_{t_0}^{t_f} dt (\text{Re}\nu_k \cos(\omega_k t) + \text{Im}\nu_k \sin(\omega_k t))(x'(t) - x(t)).
\end{aligned} \tag{2.10}$$

For practical purposes, we generally assume that the interaction began at $t_0 = 0$ and continued up until some final time t_f , so as to avoid having infinite limits in the integrals.

We will now assume that the action of the system variables is of the usual form

$$S_{\text{sys}}[x(t)] = \int_{t_0}^{t_f} L(x(t), \dot{x}(t)) dt. \tag{2.11}$$

We can then change variables to

$$X = \frac{1}{2}(x + x'), \tag{2.12a}$$

$$\xi = x - x', \tag{2.12b}$$

and write the decoherence functional in terms of the new variables. This is easily shown to be

$$\begin{aligned}
D[X(t), \xi(t)] = \exp \frac{i}{\hbar} \Big\{ & \sum_k \int_{t_0}^{t_f} dt \, \xi(t) \left[-\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{X}} \right) (X(t), \dot{X}(t)) + \frac{\partial L}{\partial X} (X(t), \dot{X}(t)) \right. \\
& + \gamma_k \sqrt{\frac{2\hbar}{m\omega_k}} (\text{Re}\nu_k \cos(\omega_k(t)) + \text{Im}\nu_k \sin(\omega_k(t))) - \frac{\gamma_k^2}{m\omega_k} \int_{t_0}^t ds \sin(\omega_k(t-s)) X(s) \Big] \\
& + \frac{i\gamma_k^2}{4m\omega_k} \int_{t_0}^{t_f} dt \int_{t_0}^t ds \cos(\omega_k(t-s)) \xi(t) \xi(s) \\
& \left. - \xi_0 \frac{\partial L}{\partial \dot{X}} (X_0, \dot{X}_0) + O(\xi^3) \right\} \chi(x'_0; x_0). \tag{2.13}
\end{aligned}$$

Note that the real part of the phase includes the Euler-Lagrange equation of motion for the system, with the addition of a retarded force due to the interaction with the reservoir. The imaginary part is strictly non-negative, with a minimum at $\xi(t) \equiv 0$, and hence tends to suppress $D[x'(t), x(t)]$ for large ξ . This makes our expansion in terms of $\xi(t)$ seem reasonable, and also causes the system to decohere, at least approximately, since $\xi(t) \neq 0$ corresponds to off-diagonal terms. The ξ_0 term occurs because of an integration by parts.

B. The Classical Equivalent.

We can now look at the classical system equivalent to the above quantum system, i.e., with the same action functional and distribution of oscillators. A coherent state is often characterized as a more “classical” state of an oscillator than the usual Fock states; it can be thought of as the state of an oscillator begun at a given initial position and momentum, within the limits imposed by the uncertainty principle.

We will begin by assuming knowledge of the trajectory of the system variable $x(t)$, and ask what the behavior of the reservoir of harmonic oscillators will be [6]. Assume that we start the oscillators in a definite state $Q^k(t = t_0) = q^k, \dot{Q}^k(t = t_0) = v^k$. The interaction potential is linear, so we can treat the trajectory of the system variable $x(t)$ as a simple driving force, giving us an equation of motion for the k th oscillator

$$\frac{d^2 Q^k}{dt^2} = -\omega_k^2 Q^k + (\gamma_k/m)x(t). \tag{2.14}$$

The solution to this equation is simply

$$Q^k(t) = q^k \cos(\omega_k(t - t_0)) + (v^k/\omega_k) \sin(\omega_k(t - t_0)) + \frac{\gamma_k}{m\omega_k} \int_{t_0}^t \sin(\omega_k(t - s))x(s)ds. \quad (2.15)$$

If the system is described by a Lagrangian $L(x, \dot{x})$, then we can write down the Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{X}}(x(t), \dot{x}(t)) - \frac{\partial L}{\partial X}(x(t), \dot{x}(t)) + \sum_k \gamma_k Q^k(t) = 0. \quad (2.16)$$

We can clearly substitute the above expression (2.15) for $Q^k(t)$ in the Euler-Lagrange equation to get

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{X}}(x(t), \dot{x}(t)) - \frac{\partial L}{\partial X}(x(t), \dot{x}(t)) + \sum_k \gamma_k \left(q^k \cos(\omega_k(t - t_0)) + (v^k/\omega_k) \sin(\omega_k(t - t_0)) \right. \\ \left. + \frac{\gamma_k}{m\omega_k} \int_{t_0}^t \sin(\omega_k(t - s))x(s)ds \right) = 0. \end{aligned} \quad (2.17)$$

This expression is clearly closely related to the real part of the phase in the decoherence functional, if we make the identity

$$\sqrt{\frac{2\hbar}{m\omega_k}} \operatorname{Re} \nu_k = q^k, \quad (2.18a)$$

$$\sqrt{\frac{2\hbar\omega_k}{m}} \operatorname{Im} \nu_k = v^k. \quad (2.18b)$$

If we write the above classical equation as $e(t) = 0$, then the real part of the phase is just

$$\int_{t_0}^{t_f} \xi(t) e(t) dt.$$

How do we interpret the imaginary part of the phase, however? In treating reservoirs in an initial thermal state, we identified this term as the effect of a stochastic force $F(t)$ arising due to thermal noise. However, in this system, there *is* no noise classically. The persistence of this term indicates a fundamental difference between the quantum and classical systems. As Gell-Mann and Hartle point out [3], in the quantum system there is always noise from zero-point oscillations, unlike classical oscillators. So the actual equation of motion derived from the quantum theory is

$$\begin{aligned}
0 &= \frac{d}{dt} \frac{\partial L}{\partial \dot{X}}(x(t), \dot{x}(t)) - \frac{\partial L}{\partial X}(x(t), \dot{x}(t)) - F(t) \\
&+ \sum_k \gamma_k \left(q^k \cos(\omega_k(t - t_0)) + (v^k/\omega_k) \sin(\omega_k(t - t_0)) - \frac{\gamma_k}{m\omega_k} \int_{t_0}^t \sin(\omega_k(t - s)) x(s) ds \right).
\end{aligned} \tag{2.19}$$

where $F(t)$ is a stochastic force with $\langle F(t) \rangle = 0$, and a two-time correlation function

$$\langle F(t)F(s) \rangle = \sum_k \frac{\gamma_k^2 \hbar}{4m\omega_k} \cos(\omega_k(t - s)). \tag{2.20}$$

C. The Continuum Limit.

In order to consider the sorts of classical systems we are concerned with, we must go to the limit of a continuum of oscillator frequencies, both classically and quantum mechanically. In doing this, we replace our sums over oscillators with integrals over a distribution function $g(\omega)$. The usual choice for such a $g(\omega)$ is the Debye distribution [6,7]:

$$g(\omega) = \begin{cases} \eta\omega^2/\Omega^2, & \omega < \Omega, \\ 0, & \omega > \Omega. \end{cases} \tag{2.21}$$

The reservoir degrees of freedom will become a continuum, $Q^k(t) \rightarrow Q(\omega, t)$, and the eigenvalues ν_k will become a continuous complex function $\nu(\omega)$. In general, Ω must be taken to be fairly large. More precisely, we want $\Omega \gg 1/(t_f - t_0)$, so that the relaxation time of the reservoir is much less than the time-scale of the problem.

Let's consider now the various components of $W[X(t), \xi(t)]$ one at a time. In the continuum limit, we have

$$\sum_k \frac{\gamma_k^2}{m\omega_k} \int_{t_0}^t ds \sin(\omega_k(t - s)) X(s) \rightarrow \frac{1}{m} \int_0^\Omega d\omega \int_{t_0}^t ds \frac{g(\omega)}{\omega} \sin(\omega(t - s)) X(s). \tag{2.22}$$

We can invert the order of integration and do the ω integral, substituting (2.21) for $g(\omega)$:

$$\begin{aligned}
\frac{1}{m} \int_{t_0}^t ds \int_0^\Omega d\omega \frac{g(\omega)}{\omega} \sin(\omega(t - s)) X(s) &= \frac{\eta}{m\Omega^2} \int_{t_0}^t ds \int_0^\Omega d\omega \omega \sin(\omega(t - s)) X(s), \\
&= \frac{\eta}{m\Omega^2} \int_{t_0}^t ds \left(-\frac{\Omega \cos(\Omega(t - s))}{t - s} + \frac{\sin(\Omega(t - s))}{(t - s)^2} \right) X(s), \\
&= \frac{\eta}{m\Omega^2} \int_{t_0}^t ds \frac{d}{ds} \left(\frac{\sin(\Omega(t - s))}{t - s} \right) X(s).
\end{aligned} \tag{2.23}$$

Now we use the fact that Ω is large. This implies that the sinusoidal terms will oscillate very rapidly, so that the integral will tend to cancel out to zero. We expect the largest contribution to come in the region where s is close to t . Thus, we expand $X(s)$ about t to get $X(s) \approx X(t) - \dot{X}(t)(t-s) + \dots$. Substituting this into the above integral, we can solve it term by term to get

$$\frac{1}{m} \int_{t_0}^t ds \int_0^\Omega d\omega \frac{g(\omega)}{\omega} \sin(\omega(t-s)) X(s) \approx -\frac{\eta}{m\Omega} X(t) - \frac{\pi\eta}{2m\Omega^2} \dot{X}(t) + O(\Omega^{-3}), \quad (2.24)$$

where the additional terms become small in the limit of a large cutoff Ω .

The second term has the form of a dissipation with constant $2\Gamma = \pi\eta/2mM\Omega^2$. The first term is a linear restoring force. If the system were a harmonic oscillator, this would cause a shift in the oscillator frequency. We can absorb this term into the system action as an additional harmonic oscillator potential:

$$S_{\text{sys}}[x(t)] \rightarrow S_{\text{sys}}'[x(t)] = S_{\text{sys}}[x(t)] - \int_{t_0}^{t_f} \frac{\eta}{2m\Omega} x^2(t). \quad (2.25)$$

If our Lagrangian is the usual $L(x, \dot{x}) = \frac{1}{2}M\dot{x}^2 - U(x)$, then we effectively have a new Lagrangian

$$L(x, \dot{x}) \rightarrow L'(x, \dot{x}) = \frac{1}{2}M\dot{x}^2 - U'(x), \quad (2.26)$$

where

$$U'(x) = U(x) + \frac{\eta}{2m\Omega} x^2. \quad (2.27)$$

In subsequent analysis, it will be $U'(x)$ that we are interested in, as the effective potential.

The imaginary part of $W[X, \xi]$ is also of interest. Here we have

$$\begin{aligned} \sum_k \frac{\gamma_k^2}{4m\omega_k} \int_{t_0}^{t_f} ds \cos(\omega_k(t-s)) \xi(t) \xi(s) &\rightarrow \frac{1}{4m} \int_0^\Omega d\omega \int_{t_0}^{t_f} ds \frac{g(\omega)}{\omega} \sin(\omega(t-s)) \xi(t) \xi(s), \\ &= \frac{\eta}{4m\Omega^2} \int_{t_0}^{t_f} ds \int_0^\Omega d\omega \omega \cos(\omega(t-s)) \xi(t) \xi(s), \\ &= \frac{\eta}{4m\Omega^2} \int_{t_0}^{t_f} ds \frac{d}{ds} \left(\frac{\cos(\Omega(t-s)) - 1}{t-s} \right) \xi(t) \xi(s). \end{aligned} \quad (2.28)$$

If we do this derivative we see

$$\frac{d}{ds} \left(\frac{\cos(\Omega(t-s)) - 1}{t-s} \right) = \frac{\Omega \sin(\Omega(t-s))}{t-s} + \frac{\cos(\Omega(t-s)) - 1}{(t-s)^2},$$

where the first term is larger than the second by a factor of roughly $\Omega(t_f - t_0)$. In the limit of large Ω this term approaches a delta function. We again expand $\xi(s)$,

$$\xi(s) \approx \xi(t) - \dot{\xi}(t)(t-s) + \dots$$

and substitute it into (2.28) to get

$$\frac{\eta}{4m\Omega^2} \int_{t_0}^{t_f} ds \frac{d}{ds} \left(\frac{\cos(\Omega(t-s)) - 1}{t-s} \right) \xi(t) \xi(s) \approx \frac{\pi\eta}{4m\Omega} \xi^2(t) + O(\Omega^{-2}). \quad (2.29)$$

We define a new constant $K = \pi\eta/2m\Omega$.

At this point, someone will likely cry foul. In the real part, we kept the two lowest order terms, while in the imaginary part we kept only one! I will give three different arguments why this is legitimate:

1. The real and imaginary parts of $W[X(t), \xi(t)]$ serve different purposes in the decoherence functional, and have different effects. A small correction to the imaginary part has only a minor effect on the level of decoherence. The inclusion of a small amount of dissipation in the real part, by contrast, leads to qualitatively different solutions.

2. The leading order term of the real part can be absorbed into the system action by going to an effective potential $U'(x)$, as we've seen, so that we must go to the next order to observe fundamental changes in the classical equation of motion.

3. The imaginary part of $W[X(t), \xi(t)]$ corresponds to a small stochastic noise. In the above limit the two-time correlation function becomes $\langle F(t)F(s) \rangle = (\pi\eta\hbar/4m\Omega)\delta(t-s) = \hbar K\delta(t-s)$. In the usual limit where \hbar is small, any corrections to the above correlation function would be too tiny to matter. The dissipative term lacks this factor of \hbar .

Finally, we have the terms arising from the initial condition of the reservoir variables. When we go to the continuum limit here, the discrete sum in (2.10) becomes an integral. If we choose $\nu(\omega) \sim \delta(\omega - \omega_0)$, then we get

$$\sum_k \gamma_k \sqrt{\frac{2\hbar}{m\omega_k}} (a_k \cos(\omega_k t) + b_k \sin(\omega_k t)) \xi(t) \rightarrow Mq \cos(\omega_0 t + \phi_0) \xi(t), \quad (2.30)$$

where q can be set arbitrarily by adjusting the amplitude of $\text{Re}\nu(\omega)$. This term has the form of a periodic driving force. We can set the phase ϕ_0 to zero by making $\text{Im}\nu(\omega) = 0$. Physically, the presence of this term is equivalent to the system being driven by a plane wave at the frequency ω_0 .

Thus, our decoherence functional becomes

$$D[X(t), \xi(t)] = \exp \frac{i}{\hbar} \left\{ \int_{t_0}^{t_f} dt \left(-M\ddot{X}(t) - \frac{dU'}{dX}(X(t)) - 2M\Gamma\dot{X}(t) + Mq \cos(\omega_0 t) \right) \xi(t) \right. \\ \left. - M\dot{X}_0 \xi_0 + iK \int_{t_0}^{t_f} dt \xi^2(t) + O(\xi^3) \right\} \chi(x'_0; x_0). \quad (2.31)$$

which gives us a quasiclassical equation of motion

$$\ddot{x} + \frac{1}{M} \frac{dU'}{dx}(x) + 2\Gamma\dot{x} = q \cos(\omega_0 t) + F(t)/M, \quad (2.32)$$

where $F(t)$ is a stochastic force with $\langle F(t) \rangle = 0$ and $\langle F(t)F(s) \rangle = \hbar K \delta(t - s)$. In a completely classical derivation, of course, this stochastic force would be absent. Thus, we have found a quantum system equivalent (in the appropriate limits) to a classical nonlinear oscillator with a periodic driving force and dissipation. All that remains now is to specialize to a chaotic example.

III. THE DAMPED, DRIVEN DUFFING OSCILLATOR

The quasiclassical equation of motion (2.32) is a fairly general expression for a one-dimensional damped, driven system. Many such systems exist which exhibit chaotic behavior for some values of the constants Γ and q . The ordinary pendulum is an example of such a system, where $U'(x) = -\cos(x)$. We will be examining another system: the damped, driven Duffing oscillator. This nonlinear oscillator has a polynomial potential

$$\frac{1}{M} U'(x) = \frac{1}{4} x^4 - \frac{1}{2} x^2. \quad (3.1)$$

We choose units to set $M = 1$. This system has the advantage of having been thoroughly studied and examined in the past, and also, since the potential is a polynomial, of not having an infinite number of nonzero derivatives. We will later see that this is convenient, though not vital.

The equation of motion is now

$$\ddot{x} + 2\Gamma\dot{x} + (x^3 - x) = q \cos(\omega_0 t) + F(t). \quad (3.2)$$

The potential is double-welled (see fig. 1). For some values of the constants, the oscillator undergoes periodic motion. By adjusting the frequency, one causes the system to undergo a series of bifurcations until eventually it enters into a region of chaotic behavior, typified by the presence of a strange attractor (see fig. 2). If one adjusts the driving force further and further the chaotic region is eventually left, and periodic motion returns [8].

It is convenient to look at the long-term chaotic behavior in terms of a *constant phase* map or *surface of section*. That is, we consider the position x and momentum p at the discrete times $t_i = 2\pi i/\omega_0$. By then plotting the values x_i and p_i we make the fractal structure of the strange attractor very clear (see fig. 3). We are also able to bring the mathematical toolbox of discrete dynamical system theory to bear on the problem. Formally, we define the constant phase map $x_i \rightarrow x_{i+1} = f_x(x_i, p_i)$, $p_i \rightarrow p_{i+1} = f_p(x_i, p_i)$, where \mathbf{f} is an operator which evolves the point (x_i, p_i) in phase space forward in time by $2\pi/\omega_0$.

We can now define a probability measure $P(x, p)$ on our phase space. In this discrete dynamics, it evolves according to the equation

$$P_{i+1}(x, p) = \int dx' \int dp' \delta(x - f_x(x', p')) \delta(p - f_p(x', p')) P_i(x', p'), \quad (3.3)$$

It is very useful then to consider an *invariant measure*, which gives a probability distribution on the strange attractor. This is defined by the equation

$$P_{i+1}(x, p) = P_i(x, p) = P_{\text{inv}}(x, p). \quad (3.4)$$

For a chaotic system such as the Duffing oscillator, P_{inv} will not be an analytic function; rather, it will be a generalized function. Also, it will not in general be unique; there are

many invariant measures, most corresponding to unstable solutions, fixed points or periodic points. It has been shown that the inclusion of a small amount of noise removes both of these objections, eliminating the unstable solutions and making the function analytic [9]. The inclusion of noise effectively broadens the δ -functions in (3.3), making it impossible for probability measure to remain poised on an unstable fixed-point solution or periodic orbit, leaving only the strange attractor as a stable set. One can therefore define the classical P_{inv} as the limit of this unique P_{inv} as the noise goes to zero; it will still be a generalized function with a fractal structure, but now unique. Note that in the quantum system, the noise is *always* non-zero; we will see how this modifies our definitions in sections IV and V.

The structure of the strange attractor arises as a limit of repeated stretching and folding of phase space. As we look closer and closer at the component points of the attractor, we see repeated layers of substructure at every scale (see fig. 4). Such infinite substructure is commonly characterized by its *fractal dimension*. There are a number of ways of defining dimension, each of which has slightly different properties.

One common definition is that of the *capacity* or *Kolmogorov dimension*, D_C . This is calculated by means of a box-counting algorithm. Phase space is divided into small cells of linear size ϵ , and one counts the number of cells $N(\epsilon)$ which contain points of the attractor. The dimension is then

$$D_C = \lim_{\epsilon \rightarrow 0} -\frac{\ln N(\epsilon)}{\ln \epsilon}. \quad (3.5)$$

Though this definition is fairly easy to calculate numerically, it does not reflect the fact that an orbit may visit regions of the attractor with varying frequency. To take this into account, one may instead use the *information dimension*, D_I . Again, phase space is divided into cells of linear dimension ϵ . The probability that a given point will fall in the i th cell is p_i . D_I is then

$$D_I = \lim_{\epsilon \rightarrow 0} \frac{\sum_i p_i \ln p_i}{\ln \epsilon}. \quad (3.6)$$

If p_i is equal for all cells that are visited, then $D_I = D_C$; otherwise $D_I < D_C$.

If one examines a small cell of phase space evolve according to our equation, it will tend to be stretched along one dimension. The overall phase space volume, however, will contract, due to the effects of dissipation. This stretching is what provides the well known signature of chaotic systems, sensitivity to initial conditions. The contraction, together with the more global process of folding, is what leads to the fractal structure of the attractor. One can average these two effects over the entire attractor to calculate the Lyapunov exponents. In our two-dimensional phase space, this will be a pair of numbers λ_1 and λ_2 , with $\lambda_1 > 0$ characterizing the stretching and $\lambda_2 < 0$ characterizing the contraction. Since overall phase space volume is shrunk by this system, clearly $\lambda_1 + \lambda_2 < 0$. If our phase space was n dimensional, there would clearly be n characteristic exponents.

Lyapunov exponents are calculated by considering the time evolution of an infinitesimal frame of basis vectors in phase space. One can perform a Gram-Schmidt orthogonalization, separating out the most rapidly increasing direction from less rapidly increasing directions repeatedly until one has n orthogonal vectors. One then takes the logarithm of the rate of change in each of those directions. Allowing the frame to evolve for many driving cycles lets one follow a phase space cell as it samples all parts of an attractor. In this way one calculates the average values of the exponents. The values of the λ_i are global quantities, characterizing the attractor as a whole, or equivalently, the long-term behavior of orbits throughout the attractor.

Calculating the highest exponent, λ_1 , is not very difficult. Finding values for a full spectrum of exponents, however, is rather tricky, and requires a subtle touch. I refer those interested to the papers of Wolf, et al. and Brown, et al. for details [9,10]. These definitions for fractal dimension and Lyapunov exponents run into trouble in the quantum case, where taking limits as $\epsilon \rightarrow 0$ is not very well-defined. We will see in section V how one can adjust these definitions appropriately.

IV. QUANTUM MAPS AND STRANGE ATTRACTORS

A. Quantum Maps.

In the classical case we went from continuous to discrete dynamics by going to the constant phase map of the Duffing oscillator. In considering a quantum equivalent, it is convenient to coarse-grain our selected trajectories $x(t)$ and $x'(t)$ by considering only their values at the times $t_i = 2\pi i/\omega_0$ of constant phase. The decoherence functional then becomes

$$D[\{x_i\}, \{x'_i\}] = \int_{\{x_i\}} \delta x \int_{\{x'_i\}} \delta x' D[x(t), x'(t)], \quad (4.1)$$

where the decoherence functional on continuous trajectories $D[x(t), x'(t)]$ is given by (2.31). The path integrals are over all paths of $x(t)$ and $x'(t)$ which pass through the points x_i and x'_i respectively at times t_i .

Such a coarse-graining is discussed by Gell-Mann and Hartle [3]. In general, in order for such a system to be sufficiently decoherent, we must also coarse-grain on the positions $\{x_i\}$ and $\{x'_i\}$. Instead of specifying the positions exactly, we instead require just that the positions fall in one of a group of short intervals $\Delta_{\alpha_i}^i$ at the times t_i . A history is then given by specifying the sequence of α_i 's. We'll use the short-hand notation α for this sequence.

We can estimate the minimum length d of such intervals by requiring that the off-diagonal terms of the decoherence functional be strongly suppressed for $|x_i - x'_i| = |\xi_i| > d$. As Gell-Mann and Hartle show, this depends on the separation between times t_i , the strength of the coupling, and so forth. A rough estimate gives

$$\Delta t \sim \frac{1}{\Omega} \exp\left(\frac{\pi\hbar}{M\Gamma d^2}\right). \quad (4.2)$$

For the quantum maps the interval is $\Delta t = 2\pi/\omega_0$. Ω is the frequency cutoff; it is equal to $K/M\Gamma$. So we get an estimate

$$d^2 \sim \frac{\pi\hbar}{M\Gamma \ln(2\pi K/M\Gamma\omega_0)}. \quad (4.3)$$

Since equation (2.31) is expressed in terms of the variables $X(t)$ and $\xi(t)$, it might be useful to change variables in our coarse-grained systems. As the decoherence functional is

suppressed for large ξ , we can treat our integrand as being quadratic in $\xi(t)$, and carry out the ξ integration. This gives us

$$p(\alpha) = \sqrt{\frac{2\pi}{K}} \int_{\{\alpha_i\}} \delta X \exp\left\{-\frac{1}{K\hbar} \int_{t_0}^{t_f} e^2(t) dt\right\} w(X_0, p_0), \quad (4.4)$$

where $e(t) = 0$ is the classical equation of motion as given above in (2.32) and $w(X_0, p_0)$ is the initial Wigner distribution, obtained by the integral over ξ_0 .

We see that if the $\{X_i\}$ do not lie along a classical trajectory $e(t) = 0$, then the functional will be suppressed. So the most probable histories are those which lie along the classical trajectory. The X path integral cannot be done exactly in most cases, but one can see that in general the $\{X_i\}$ must lie near the $\{x_i\}$ for some classical problem for the probability to be of reasonable magnitude.

The Wigner distribution $w(X, p)$ (not to be confused with the influence phase $W[X(t), \xi(t)]$, which is a functional!) is given by

$$w(X, p) = \frac{1}{\pi} \int_{-\infty}^{+\infty} e^{i\xi p/\hbar} \tilde{\rho}(X + \xi/2, X - \xi/2) d\xi, \quad (4.5)$$

where we see that X and ξ are our usual variables, and p has units of momentum. $w(X, p)$ is somewhat analogous to a probability distribution on phase space. It is real, and integrates to a total of 1; its primary difference from a classical phase space distribution is that it is not strictly non-negative.

If we want to advance $w(X, p)$ in time, we can define a transfer matrix \mathbf{T} such that $w(t_f) = \mathbf{T}w(t_0)$. More explicitly this is

$$w(X_f, p_f) = \int dX_0 \int dp_0 T(X_f, p_f; X_0, p_0) w(X_0, p_0) \quad (4.6)$$

where $T(X_f, p_f; X_0, p_0)$ is defined

$$T(X_f, p_f; X_0, p_0) = \frac{1}{\pi} \int d\xi_0 d\xi_f e^{i(\xi_f p_f - \xi_0 p_0)/\hbar} \tilde{T}(X_f + \xi_f/2, X_f - \xi_f/2; X_0 + \xi_0/2, X_0 - \xi_0/2), \quad (4.7a)$$

$$\tilde{T}(x_f, x'_f; x_0, x'_0) = \int \delta x \delta x' \exp \frac{i}{\hbar} \left\{ S_{\text{sys}}[x(t)] - S_{\text{sys}}[x'(t)] + W[x(t), x'(t)] \right\} \quad (4.7b)$$

If we let $t_f - t_0 = 2\pi/\omega_0$ then time advancement can be performed by repeated applications of \mathbf{T} . This is a sort of *quantum map*,

$$w_i \rightarrow w_{i+1} = \mathbf{T}w_i. \quad (4.8)$$

We can ask if repeated applications of \mathbf{T} will tend to converge to some invariant Wigner distribution $w_{\text{inv}} = \mathbf{T}w_{\text{inv}}$, analogous to the invariant measure P_{inv} of section III. Preliminary numerical calculations seem to show that this is the case [11]. This w_{inv} appears unique, and should be analytic, thanks to the “blurring” effect of quantum noise.

To make closer contact with the classical system, we might wish instead to consider histories in which a trajectory passes through small cells in phase space, rather than just intervals in X . We can write such a history by considering projections onto intervals in X followed very briefly by projections onto intervals in p . Histories of this type have been considered by Gell-Mann and Hartle, and by Halliwell, who wrote down an explicit equation for such a history [3,12].

One cannot in general specify both the momentum and position of a particle at the same instant. One can, however, consider a measurement of position followed by a measurement of momentum, and let the time between them go to zero. Halliwell calculated the probability of such a history using approximate projections

$$P_{\bar{x}} = \frac{1}{\pi^{1/2}\sigma_{\bar{x}}} \int_{-\infty}^{\infty} dx \exp\left[-\frac{(x - \bar{x})^2}{\sigma_{\bar{x}}^2}\right] |x\rangle\langle x|, \quad (4.9a)$$

$$P_{\bar{p}} = \frac{1}{\pi^{1/2}\sigma_{\bar{p}}} \int_{-\infty}^{\infty} dp \exp\left[-\frac{(p - \bar{p})^2}{\sigma_{\bar{p}}^2}\right] |p\rangle\langle p|. \quad (4.9b)$$

These are approximate projections into intervals $\Delta_{\bar{x}}$ of width $\sigma_{\bar{x}}$ and $\tilde{\Delta}_{\bar{p}}$ of width $\sigma_{\bar{p}}$, centered on \bar{x} and \bar{p} , respectively. Halliwell shows [12] that for an initial Wigner distribution $w(X, p)$, the probability of finding a particle in the phase-space cell delimited by the two above projections is

$$p(\Delta_{\bar{x}}, \tilde{\Delta}_{\bar{p}}) = \int dX dp w(X, p) \exp[-a(p - \bar{p})^2 - b(X - \bar{x})^2], \quad (4.10)$$

where

$$a = \frac{\sigma_x^2}{2[\hbar^2 + (1/4)\sigma_x^2\sigma_p^2]}, \quad b = \frac{2}{\sigma_x^2} \quad (4.11)$$

when the X projection precedes the p projection, and

$$a = \frac{2}{\sigma_p^2}, \quad b = \frac{\sigma_p^2}{2[\hbar^2 + (1/4)\sigma_x^2\sigma_p^2]} \quad (4.12)$$

when the p projection precedes the X projection. There is a restriction on these projections that

$$0 < ab \leq \frac{1}{\hbar} \implies \sigma_x^2\sigma_p^2 > \hbar^2. \quad (4.13)$$

To calculate the probabilities of an orbit passing through a series of such cells $(\Delta_i, \tilde{\Delta}_i)$ at times t_i we make use of the transition matrix \mathbf{T} . Let us assume that the X projection comes first. Then it turns out that

$$\begin{aligned} p(\{\Delta_i, \tilde{\Delta}_i\}) &= \int d\{X\} d\{p\} w(X_0, p_0) \\ &\times \exp\left[-\frac{2}{\sigma_p^2}(p_1 - \bar{p}_0)^2 - \frac{2}{\sigma_x^2}(X_0 - \bar{x}_0)^2 - \frac{\sigma_x^2}{2\hbar^2}(p_0 - p_1)^2 - \frac{\sigma_p^2}{2\hbar^2}(X_0 - X_1)^2\right] \\ &\times T(X_2, p_2; X_1, p_1) \\ &\times \exp\left[-\frac{2}{\sigma_p^2}(p_3 - \bar{p}_1)^2 - \frac{2}{\sigma_x^2}(X_2 - \bar{x}_1)^2 - \frac{\sigma_x^2}{2\hbar^2}(p_2 - p_3)^2 - \frac{\sigma_p^2}{2\hbar^2}(X_2 - X_3)^2\right] \\ &\times T(X_4, p_4; X_3, p_3) \\ &\times \dots \end{aligned} \quad (4.14)$$

where the interval Δ_i is centered on \bar{x}_i and $\tilde{\Delta}_i$ on \bar{p}_i . The final projection will be of the form (4.10). The case where the p projection precedes the x is very similar to the above.

There are other ways of considering phase space projections, using coherent states or Gaussian combinations of coherent states. A fuller discussion of phase space histories and their decoherence deserves a fuller discussion elsewhere [13].

B. The Master Equation.

Another common method of studying systems such as this is by means of a Master equation formalism. Caldeira and Leggett derive such an equation in the case of a harmonic oscillator interacting with a thermal bath at relatively high temperature [7]. Their result is readily adapted to the present case, yielding the equation

$$\begin{aligned} \frac{\partial \tilde{\rho}}{\partial t}(x, x') = & -\frac{K}{\hbar}(x - x')^2 \tilde{\rho} + \frac{iq}{\hbar}(x - x') \cos(\omega_0 t) \tilde{\rho} - \frac{i}{\hbar}(V(x) - V(x')) \tilde{\rho} \\ & + 2\Gamma(x - x') \left(\frac{\partial \tilde{\rho}}{\partial x'} - \frac{\partial \tilde{\rho}}{\partial x} \right) + \frac{i\hbar}{2m} \left(\frac{\partial^2 \tilde{\rho}}{\partial x^2} - \frac{\partial^2 \tilde{\rho}}{\partial x'^2} \right) \end{aligned} \quad (4.15)$$

In examining this equation, the meanings of the different terms are highly intuitive. The K/\hbar term is a diffusive effect resulting from the quantum noise; the Γ term includes the effects of dissipation; the $q \cos(\omega_0 t)$ is the driving force.

Changing to the variables X and ξ , the Master equation becomes

$$\begin{aligned} \frac{\partial \tilde{\rho}}{\partial t}(X, \xi) = & -(4K/\hbar)\xi^2 \tilde{\rho} + (2iq/\hbar)\xi \cos(\omega_0 t) \tilde{\rho} - \frac{i}{\hbar}(V(X + \xi/2) - V(X - \xi/2)) \tilde{\rho} \\ & - 4\Gamma\xi \frac{\partial \tilde{\rho}}{\partial \xi} + \frac{i\hbar}{2m} \frac{\partial^2 \tilde{\rho}}{\partial \xi \partial X}. \end{aligned} \quad (4.16)$$

If ξ is small, then we can expand the potential term to give us

$$V(X + \xi/2) - V(X - \xi/2) \approx \xi \frac{\partial V}{\partial X}(X) + \frac{\xi^3}{24} \frac{\partial^3 V}{\partial X^3} + \dots \quad (4.17)$$

For the Duffing potential, of course, the higher-order terms vanish. This is a convenient benefit of dealing with a polynomial potential.

Transforming this equation by (4.5) gives us a new equation for the evolution of the Wigner distribution itself:

$$\begin{aligned} \frac{\partial w}{\partial t}(X, p) = & \hbar K \frac{\partial^2 w}{\partial p^2} - q \cos(\omega_0 t) \frac{\partial w}{\partial p} + \frac{\partial V}{\partial X}(X) \frac{\partial w}{\partial p} \\ & - \frac{\hbar^2}{24} \frac{\partial^3 V}{\partial X^3}(X) \frac{\partial^3 w}{\partial p^3} + 4\Gamma \frac{\partial}{\partial p} p w - \frac{p}{m} \frac{\partial w}{\partial X}. \end{aligned} \quad (4.18)$$

This is almost exactly the form of the Fokker-Planck equation for the classical equation of motion (2.32), with the diffusive $\hbar K$ term representing the effects of the random fluctuations

on the “probability” distribution and the third-derivative term being a purely quantum-mechanical addition, enabling $w(X, p)$ to become negative in limited regions of phase space. To interpret this distribution as a probability, we must coarse-grain by averaging it over small volumes of phase space, producing the sort of “smeared” Wigner distribution discussed by Halliwell [12].

This is not, of course, the full story. In order to correctly describe this system, we need not only the time evolution of the Wigner distribution, but also to specify a set of decoherent histories, as discussed in the previous section. Without those histories, it is impossible to assign classical probabilities in a consistent manner. These two approaches can be made to complement each other, however, as the Master equation can be solved to yield the transfer matrix \mathbf{T} , defined in the previous section as a path integral. In the case of chaos, one can in general only solve these equations numerically, and the Master equation formalism then has a computational advantage over the path integral form.

V. INTERPRETATION OF CLASSICAL QUANTITIES

From the previous section, we see that the behavior of a system such as we are examining can be evaluated on many levels:

1. The Classical level. In the previous section we saw that all histories which deviate too far from the classical solution have their probabilities highly suppressed. If a system is large enough in scale, with enough inertia that the quantum effects are lost in other sources of uncertainty, we can treat it as approximately classical. Clearly, in a chaotic system this quantum noise does cause large alterations in the overall behavior of the system, but it is often impossible to separate this from thermal noise or other sources of error.

2. The Quasiclassical level. Here we again treat the system as essentially classical, but now explicitly include the noise arising from quantum effects, which is large enough to be noticed on the scale of the system; this is the system as described by equation (2.32). From a practical point of view, this is the level at which quantum effects are most easily calculated.

This also overlaps the considerable work that has been done on dynamical systems with noise [9,14–16]. Note that this is *not* the same as a semiclassical approximation, such as WKB.

3. The Quantum level. Fundamentally, we can consider the system in terms of coarse-grainings and decoherent histories. Instead of treating a system as basically classical with added noise, we consider all possible histories, and compute expectation values for classical quantities from the probabilities of those histories.

We’ve already discussed the classical (level 1) definitions of the lyapunov exponents and fractal dimensions used to characterize chaotic systems and strange attractors in section III. As pointed out, these quantities are usually defined at least formally by calculating a quantity for the system at different levels of coarse-graining (i.e., different box sizes ϵ), and taking the limit as we go to finer and finer scales. While this has great mathematical power and consistency, in actual physical systems it inevitably breaks down. As Benoit Mandelbrot wrote on the problem of measuring coastlines with seemingly infinite levels of detail [17],

“To obtain a [fractal] Koch curve, the cascade of smaller and smaller new promontories is pushed to infinity, but in Nature every cascade must stop or change character. While endless promontories may exist, the notion that they are self-similar can apply only between certain limits. Below the lower limit, the concept of coastline ceases to belong to geography.

“It is therefore reasonable to view the real coastline as involving two *cutoff scales*. Its *outer cutoff* Ω might be the diameter of the smallest circle encompassing an island, or perhaps a continent, and the *inner cutoff* ϵ might be the twenty meters mentioned. . . Actual numerical values are hard to pinpoint, but the need for cutoffs is unquestionable.”

As we shall see, in the case of chaotic strange attractors, the underlying quantum physics effectively provides that lower cutoff.

A. Lyapunov Exponents.

Classically, the Lyapunov exponents characterize the rate at which nearby trajectories diverge as they evolve according to the equations of motion. In a chaotic system, one expects any two trajectories, no matter how close they start, to eventually move on the strange attractor completely independently of each other. This is measured in the classical case by taking the limit as points start arbitrarily near each other and evolve for arbitrarily long lengths of time.

When we allow for the presence of quantum effects, however, this definition is no longer meaningful. As points begin closer and closer to each other, the effects of noise become larger and larger; one would expect the largest exponent to diverge in the limit as $\epsilon \rightarrow 0$. As we saw in section IV, the phase space cells in a decoherent history cannot be smaller than a certain size. This limit provides the lower cutoff mentioned above.

For most systems it is impossible to calculate the values of Lyapunov exponents exactly. Instead, one performs a numerical calculation. It is easiest to calculate the highest exponent; lower exponents are more difficult, as their effects tend to be swamped by λ_1 . In a numerical calculation small errors are unavoidable; each such error will add a small admixture of the most rapidly growing component, which will quickly drown out other effects.

Because of this, we'll first consider only the value of λ_1 . A simple way of estimating λ_1 is to numerically integrate equation (2.32) for a longer period of time, to generate a large number of points $\{x_i, p_i\}$ in phase space. One can then locate nearby points, closer than a certain cutoff ϵ , and trace their trajectories until they diverge further than an upper cutoff Δ . One then calculates the logarithm of the average divergence rate and averages it over many such pairs of points.

I have calculated this quantity in the quasiclassical case (see fig. 4). It turns out that the result one calculates is not very sensitive to the upper cutoff Δ , but is highly sensitive to the lower cutoff ϵ . In figure 4 we see the measured value of λ_1 as a function of ϵ for several different relative strengths of the quantum noise $\hbar K$. Notice how for $\hbar K > 0$ λ_1 diverges as

$\epsilon \rightarrow 0$.

Because of dissipation, the overall phase-space volume of an initial distribution tends to decrease with time. This indicates that, classically, $\lambda_1 + \lambda_2 < 0$. At very small length scales, however, the effects of noise counteract the effects of dissipation, causing phase-space volume to grow rather than shrink. Thus, at small length scales we expect to see the sum $\lambda_1 + \lambda_2$ become positive, and eventually approach $\lambda_1/\lambda_2 \approx 1$; the dimension at that length scales should also approach an integer (2 in this case).

We can try to define a quantum-mechanical analog of λ_1 . While I am not sure exactly what form such a definition should take, I can make a conjecture. Suppose that we start from the invariant Wigner distribution w_{inv} at $t = t_0$. We divide phase space into small cells $\{c_i\}$, centered on average positions $\{\bar{v}_i\}$ in phase space. These cells have a characteristic size ϵ (or area ϵ^2). Let d_{ij} be the distance between the centers of the i th and j th cells. We can define p_i to be the probability that the system is in cell c_i at time t_1 , using equation (4.10), and p_{ij} to be the probability that the system is in c_i at time t_1 and c_j at time t_2 , as shown in (4.14). Clearly

$$\sum_i p_i = 1, \quad \sum_j p_{ij} = p_i.$$

The probability of the system being in c_j at time t_2 *given* that it was in c_i at t_1 is

$$p(j|i) = \frac{p_{ij}}{p_i}. \quad (5.1)$$

A rough estimate of the rate of spreading is then given by

$$\lambda(\epsilon)_{\text{qm}} = \frac{1}{2} \sum_i p_i \log \left[\frac{\frac{1}{2} \sum_{j,k} p_{ij} p_{ik} d_{jk}^2}{p_i^2 \epsilon^2} \right]. \quad (5.2)$$

It isn't clear whether this will agree with the usual definition of λ_1 in the limit. This is much more a rate of expansion averaged over the attractor, whereas λ_1 is usually defined as the rate at which nearby solutions diverge when followed for a long period of time. This latter definition has serious problems in the quantum case, where it is impossible to start solutions arbitrarily close together, and hence equally impossible to follow them

for arbitrarily long periods of time without global processes (such as folding) becoming important. When quantum effects are very small one can approach this long-orbit definition, but in that case one is really doing a quasiclassical calculation (like the one above in figure 4), where the system can be treated as a classical stochastic equation.

Numerical experiments might serve to explore the connections, if any, between these classical and quantum ideas of Lyapunov exponents. I hope to do more such exploration soon. Also, it is not clear to me exactly what form quantum equivalents to lower Lyapunov exponents might take, nor even if such a concept is useful. These questions will soon, I hope, have at least tentative answers.

B. Information Dimension.

The information dimension is, as we saw in section IV, another number used to characterize strange attractors. It is usually defined by a box-counting algorithm of the type given in (3.6). As mentioned before, when one takes quantum mechanics into account, allowing the size of a box to go to zero no longer makes much sense.

Instead, let us consider the information dimension D_I as a function of box-size:

$$D_I(\epsilon) = \frac{\sum_i p_i \ln p_i}{\ln \epsilon}. \quad (5.3)$$

The probability p_i is defined as before. For non-zero ϵ , the exact value of $D_I(\epsilon)$ will vary slightly depending on how the boxes are chosen. This ambiguity can be eliminated by taking $D_I(\epsilon)$ to be the minimum possible value over all possible arrangements of boxes. In practice, this makes little difference. The usual classical limit is then just the limit of $D_I(\epsilon)$ as $\epsilon \rightarrow 0$.

Figure 5 shows the calculated values of $D_I(\epsilon)$ for different values of $\hbar K$. As we see, at large length scales the fractal nature of the attractor is not readily apparent; as we shrink our scale, the dimension decreases, until when dropping below the lower cutoff given by the quantum effects it abruptly turns upward again. This was calculated quasiclassically, using a long orbit with associated noise.

One can also do an analogous calculation using the complete quantum theory. Consider the invariant Wigner distribution $w_{\text{inv}}(X, p)$, defined in section IV. We can define the information dimension $D_I(\epsilon)_{\text{qm}}$ using the same definition (5.3). We divide phase space into evenly-sized cells $\{c_i\}$ of size ϵ , just as in the discussion of $\lambda(\epsilon)_{\text{qm}}$ above, and use the expression (4.10) for the probability p_i of being in the cell c_i . Again, we can eliminate ambiguity by minimizing $D_I(\epsilon)_{\text{qm}}$ over all possible divisions into cells. Clearly such a dimension will not even be well defined for cells of volume less than \hbar , and will in general depend on the scale of the coarse-graining, just as in the quasiclassical treatment.

Needless to say, it is much easier to extend the information dimension D_I to a probabilistic theory than it is to find an analogy for the capacity dimension D_C . The presence of noise will give a small but non-zero probability of finding a point in any cell, even if it is far from the classical strange attractor. So, for quantum dissipative chaos at any rate, D_I seems to be the more useful quantity.

VI. QUANTUM CHAOS

Since the discovery of chaos in the 1970's, there have been numerous attempts to look for the existence of chaos in quantum mechanical systems. Almost all of these have concentrated on quantized versions of non-integrable Hamiltonian systems. A seeming paradox was at the heart of the debate: chaos, as a classical phenomenon, depended entirely on the existence of nonlinear terms in the equations of motion; yet quantum systems are by their nature completely linear.

In fact, this argument is clearly invalid. While it is true that the linearity of the Schrödinger equation and its relativistic generalizations implies that one would not expect chaotic behavior in the wave function itself, this has little bearing on what one would actually see if one observed such a system. One does not measure wave functions; one measures particles.

An analogous classical treatment would be to consider probability distributions in phase

space rather than values of position and momentum. One can then go from a set of nonlinear ordinary differential equations to a Fokker-Planck partial differential equation. The P.D.E. is completely linear. Does this then imply that chaos cannot exist in classical mechanics? Such a conclusion would be absurd.

Of course, wave functions are not probability distributions, so the comparison is a bit misleading. A sufficiently coarse-grained Wigner distribution, however, can be made to look very much like a probability distribution, and its Master equation closely resembles the Fokker-Planck equation, as we have seen, so comparing the two is not completely inappropriate.

In fact, certain quantum models can exhibit chaos [18], but they are exceptional. Most systems which occupy a bounded volume in phase space do eventually exhibit *quantum recurrence* in which the expectation values of quantities such as energy are almost periodic [19–22]. A simple information-theory argument can be made for why this should be: a bounded volume in phase space V represents a finite number of possible states $N \sim V/\hbar$, so one would expect the long-term behavior to be periodic or quasiperiodic. According to Ehrenfest’s theorem, a narrow wave packet will tend to follow the classical chaotic trajectory for a time t_E until it has spread out to a size comparable to the phase space volume. Thus, it is argued, one should observe a long chaotic “transient,” ending ultimately in periodic or almost periodic motion. As the system becomes more “macroscopic” the phase space volume becomes larger with respect to \hbar and t_E becomes longer.

Chirikov, Izrailev and Shepelyansky describe the usual approach to quantum chaos [18]. They separate the problem into two parts, the dynamics of the undisturbed wave function and the effects of measurement and wave function collapse. Obviously, this periodic behavior of expectation values says nothing about what an experimenter would actually observe upon measuring the system. They compare the former with deterministic behavior and the latter with randomness and “noise”. Clearly, an actual series of measurements would not be periodic at all, but would instead resemble a random chaotic trajectory. They further dismiss the study of dissipative chaos as a mere phenomenological approximation to an

underlying Hamiltonian system, e.g., in our case including both the system and reservoir degrees of freedom.

If the work of Gell-Mann, Hartle, and others is to be believed, we should consider only decoherent histories. Since Hamiltonian systems almost by definition do not interact with outside degrees of freedom, one cannot really talk of “measuring” them. If measurements are taken, the system is disturbed; if measurements are not taken, the system evolves undisturbed, but detailed histories of the motion will not decohere. In dissipative systems, by contrast, the chaotic system is interacting continually with the neglected degrees of freedom of the reservoir. These serve to provide a continual “measurement” of the system, in addition to causing dissipation and noise, so that histories of the system variables do decohere, as we have seen. Thus, considering these dissipative systems from a decoherence functional point of view is entirely appropriate. Also, any description of real macroscopic systems must allow for coarse-graining over the many degrees of freedom of which we are ignorant. As one goes from the classical to the quantum realms, deterministic uncertainty or randomness is replaced by probabilistic uncertainty.

In the Duffing oscillator model studied here, an arbitrary initial distribution w_0 will tend to converge onto the invariant distribution, $\lim_{n \rightarrow \infty} \mathbf{T}^n w_0 \rightarrow w_{\text{inv}}$ [11]. This invariant distribution is periodic, with period $2\pi/\omega_0$, so that $\mathbf{T}w_{\text{inv}} = w_{\text{inv}}$. This resembles the idea of a long chaotic “transient” leading to a non-chaotic, periodic behavior.

This is not, however, very different from the behavior of classical systems evolving according to the Fokker-Planck equation. Most initial distributions P_0 of non-zero width evolve into the invariant probability measure P_{inv} in much the same way that w evolves into w_{inv} in the quantum case (see fig. 6). The classical case is complicated by the non-analyticity and non-uniqueness of P_{inv} , so that there can be a finite probability of sitting on top of some unstable equilibrium or periodic point. An initial distribution with non-zero width will never actually become the invariant measure, of course, always itself remaining an analytic function if it begins as one, but the difference rapidly becomes too small to measure. These difficulties do not exist in the quantum case, due to the presence of noise. This difference,

plus the necessity for coarse-graining (which requires an initial distribution to have non-zero width, thanks to the uncertainty principle), is what distinguishes the classical and quantum cases, not the periodicity of the solution per se. The important idea is not the behavior of w_{inv} , but rather the probabilities of different possible histories of the system as described in (4.14).

Other authors have concentrated on the rate of expansion of uncertainty in quantum chaotic systems, and on how chaos leads to a form of “dissipation” in the quantum wave function itself [23–25].

A very interesting suggestion arose in work by Weinberg on possible nonlinear generalizations of quantum mechanics [26,27]. With true nonlinearity, chaos on the level of the wave function could exist. Weinberg points out, however, that any nonlinearity in the theory must be very small to be consistent with experiment, and chaos is rare in such nearly-integrable systems. While such a generalization of quantum theory is certainly not ruled out by experiment, it poses a number of troubling problems. As pointed out by Polchinski [28], either faster-than-light transmission of information via an EPR-type experiment becomes possible, or different branches of histories can continue to interact with each other, effectively making decoherence impossible. Both of these, while certainly not impossible, have absolutely no basis in experimental evidence; indeed, the latter would imply that even classical probabilities could undergo a sort of “interference” with each other.

A few papers have been published on quantum dissipative chaos [29,30], mostly by including a phenomenological term in the Schrödinger equation to model dissipation. While there is a fair bit of other work on the phenomenon of quantum dissipation itself [7,31–33], little of it has been applied to chaotic systems, as far as I am aware, and relatively little has been from the decoherence point of view.

I cannot do a broad survey of an extremely active field here. A number of good books now exist on quantum Hamiltonian chaos [34,35].

VII. CONCLUSIONS

In this paper, I have only treated a single model in depth: the forced, damped Duffing oscillator. It is clear, though, that the techniques used are easily applied to any nonlinear oscillator problem. I believe that, in general, the formalism of Gell-Mann and Hartle provides a rigorous method for treating any classical chaotic system.

For some systems, of course, such a treatment may be inappropriate. Dissipative chaos was first discovered in attempts to model fluid dynamics; while in principle such problems could be treated in this way, any quantum effects are likely to be so small as to be irrelevant. For nonlinear oscillators, though, the application is quite reasonable, and it is possible that experiments could be done in quantum optical or electronic systems which would correspond to classically chaotic systems of this sort.

What is more, using this formalism provides a link between a quantum system and its “classical limit” which lets one define the idea of quantum chaos in a rigorous way. And such systems can be treated not only semi-classically, but with the full quantum laws as well.

Equally important, in this quasiclassical treatment we can see how chaotic systems can serve to amplify the effects of small quantum fluctuations. Sensitive dependence on initial conditions – the hallmark of chaos – is an important idea in understanding *measurement situations*, in which minor quantum effects can become correlated with a change in macroscopic variables. In complex systems with many degrees of freedom, sensitive dependence on initial conditions is probably the rule rather than the exception.

A great deal remains to be done. This formalism can readily be applied to a number of other systems besides the Duffing oscillator. Also, it would be valuable to develop numerical programs for solving the Master equation, and calculating the quantities characterizing the attractor in the full quantum system as well as the quasiclassical limit. Some work on this has already been done [11], but it is still in a rather crude state. But the basic outlines of the theory are clear. Using the theory of Gell-Mann and Hartle, a rigorous treatment of

quantum dissipative chaos is finally possible.

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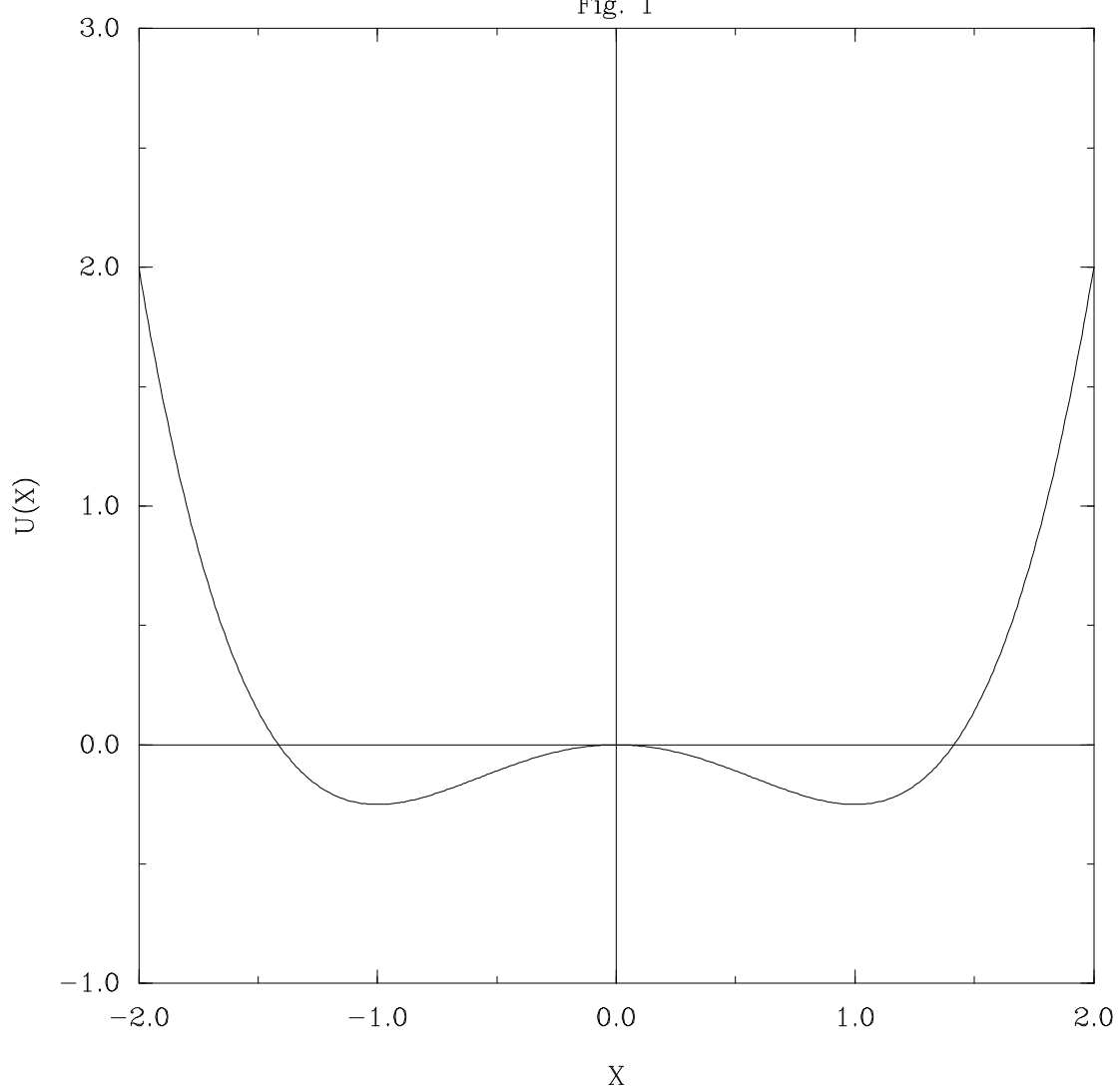
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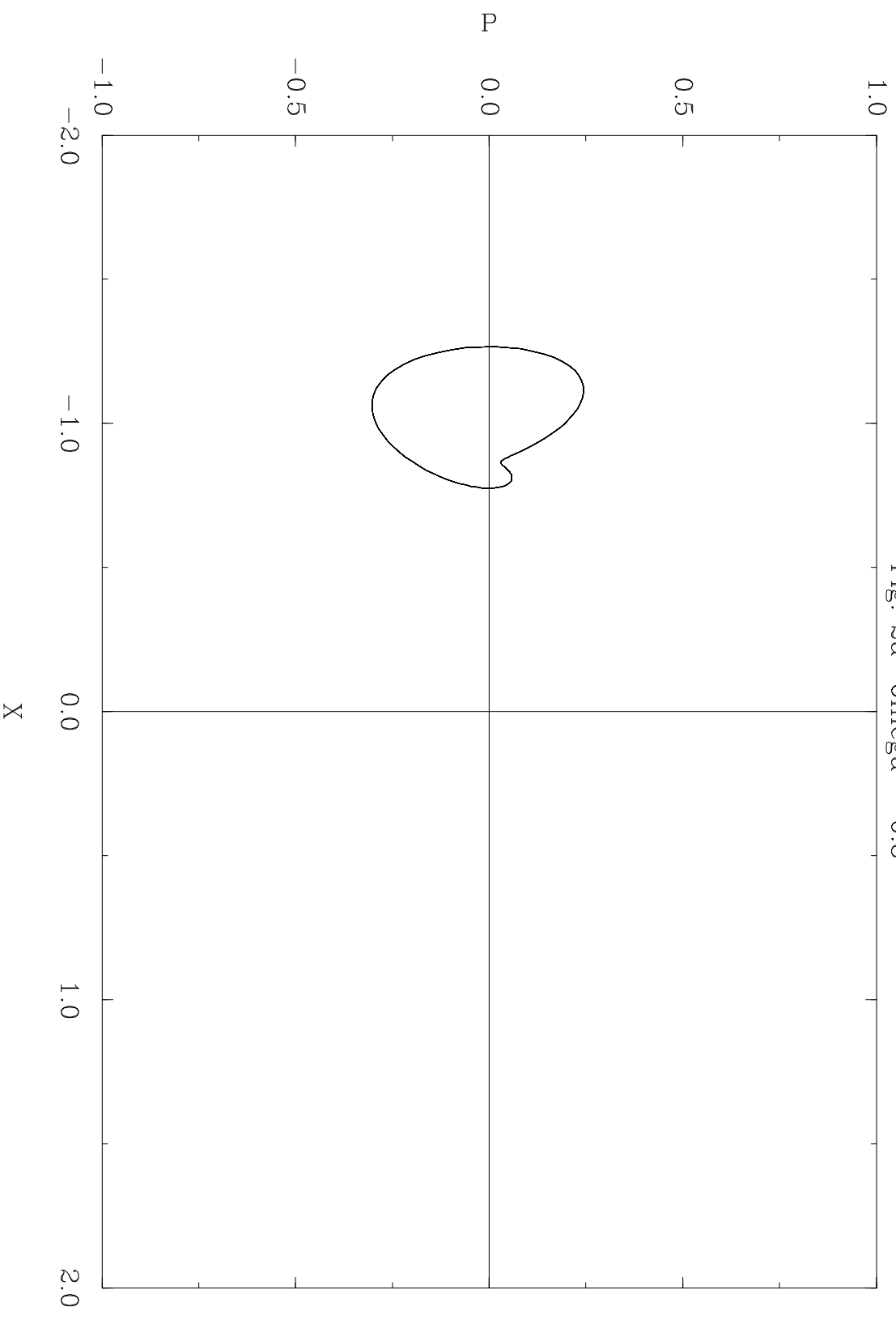
Duffing Oscillator Potential

Fig. 1



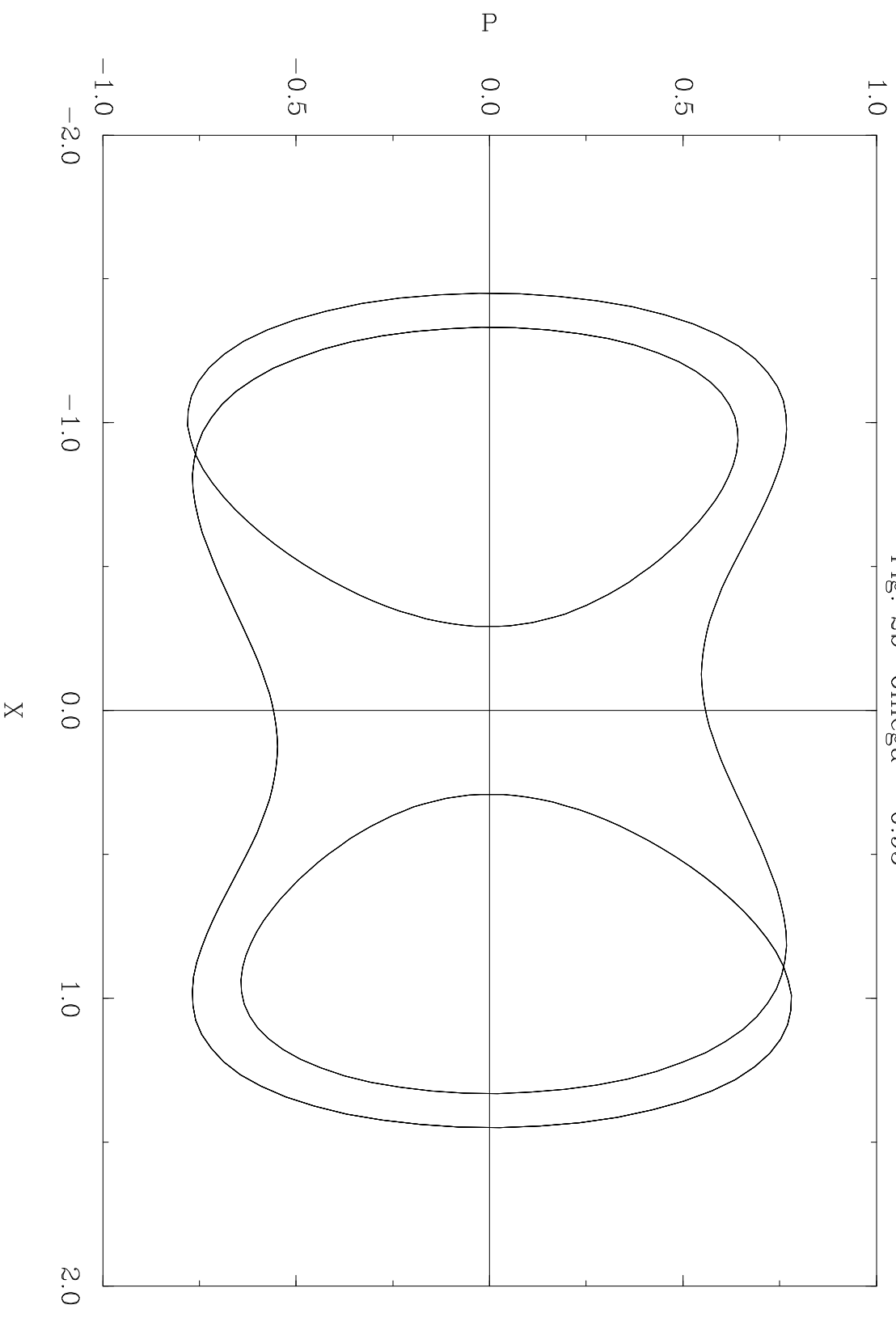
Damped, Driven Duffing Oscillator

Fig. 2a $\omega = 0.8$



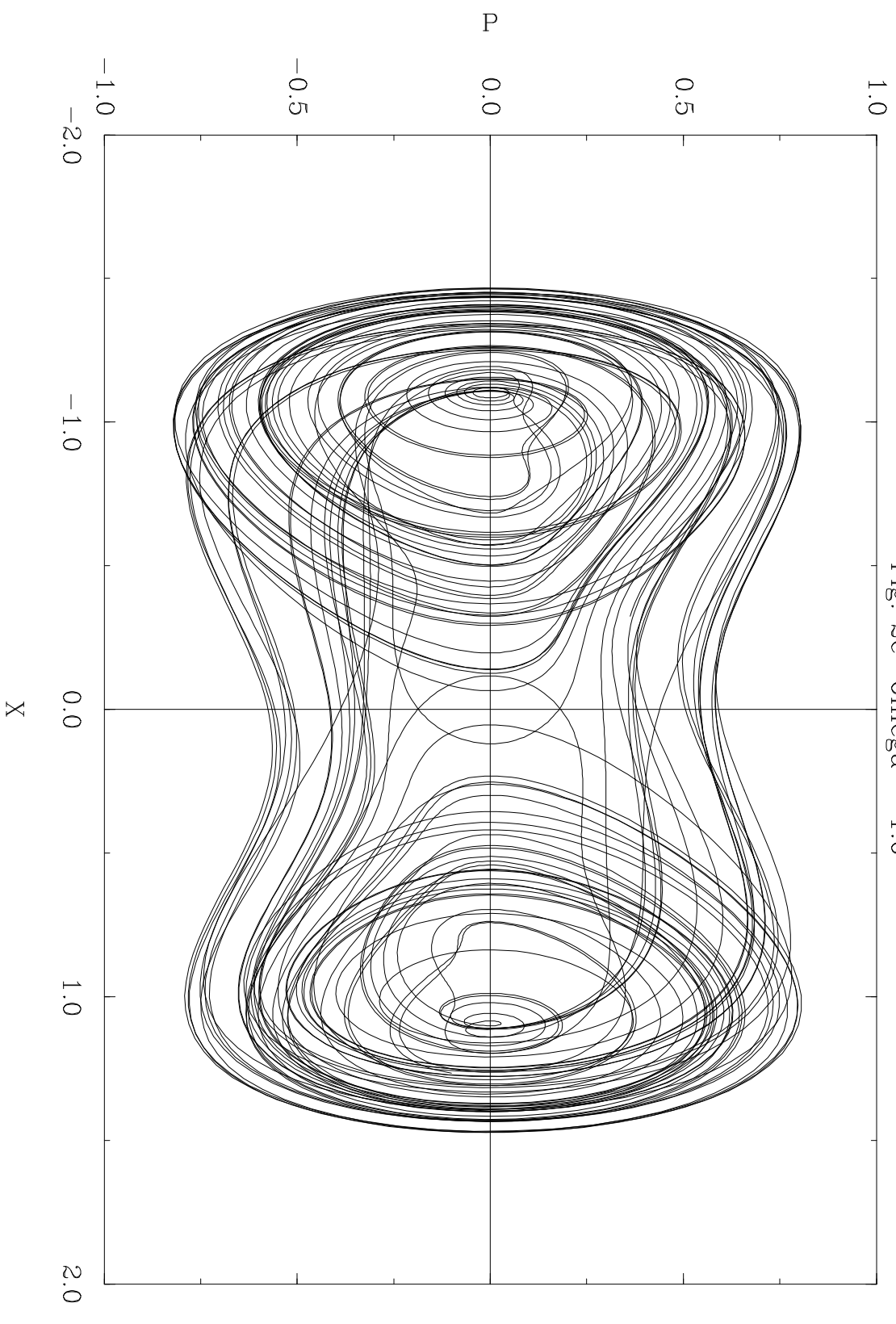
Damped, Driven Duffing Oscillator

Fig. 2b $\omega = 0.95$



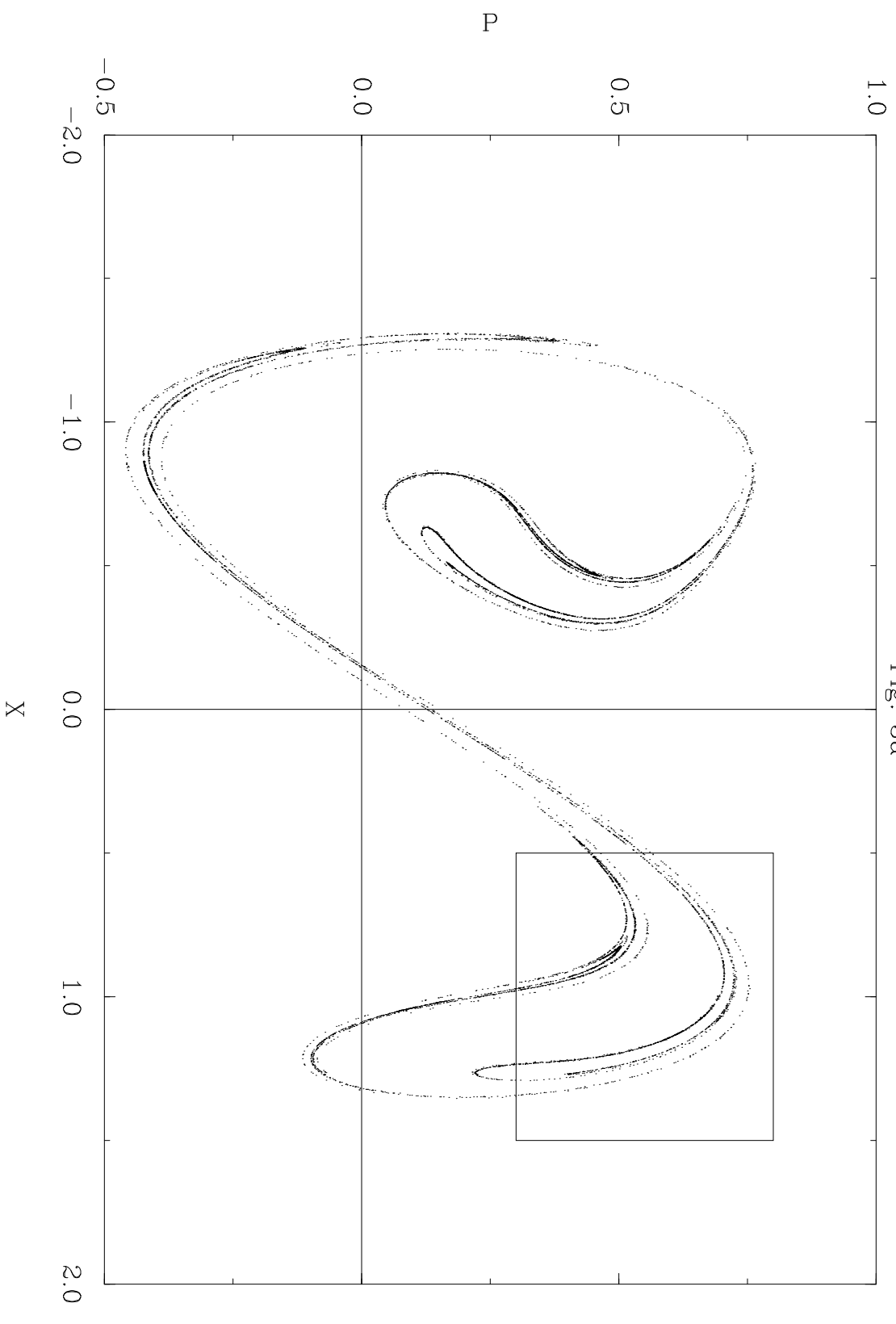
Damped, Driven Duffing Oscillator

Fig. 2c $\omega = 1.0$



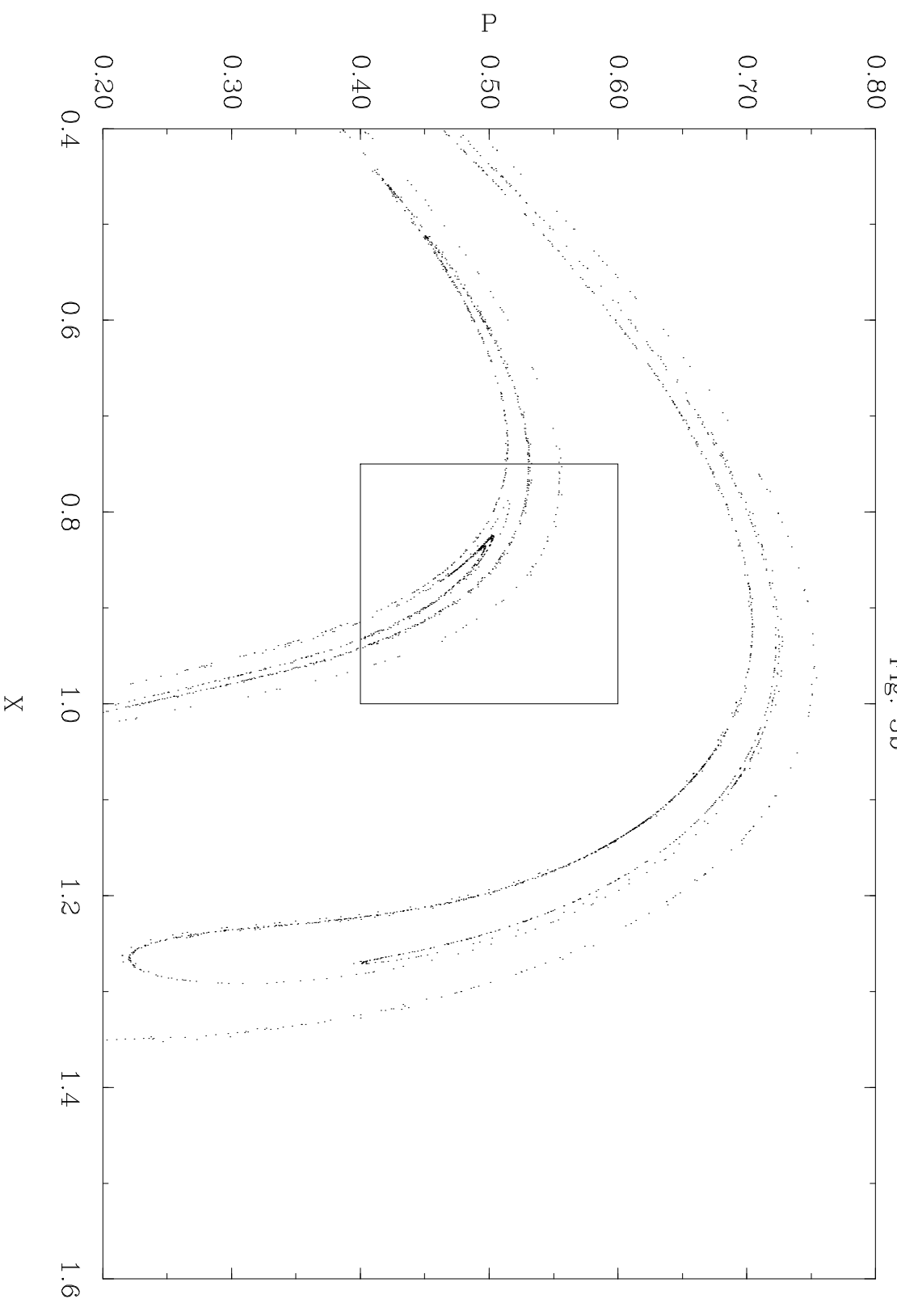
Poincare Section of Duffing Attractor

Fig. 3a



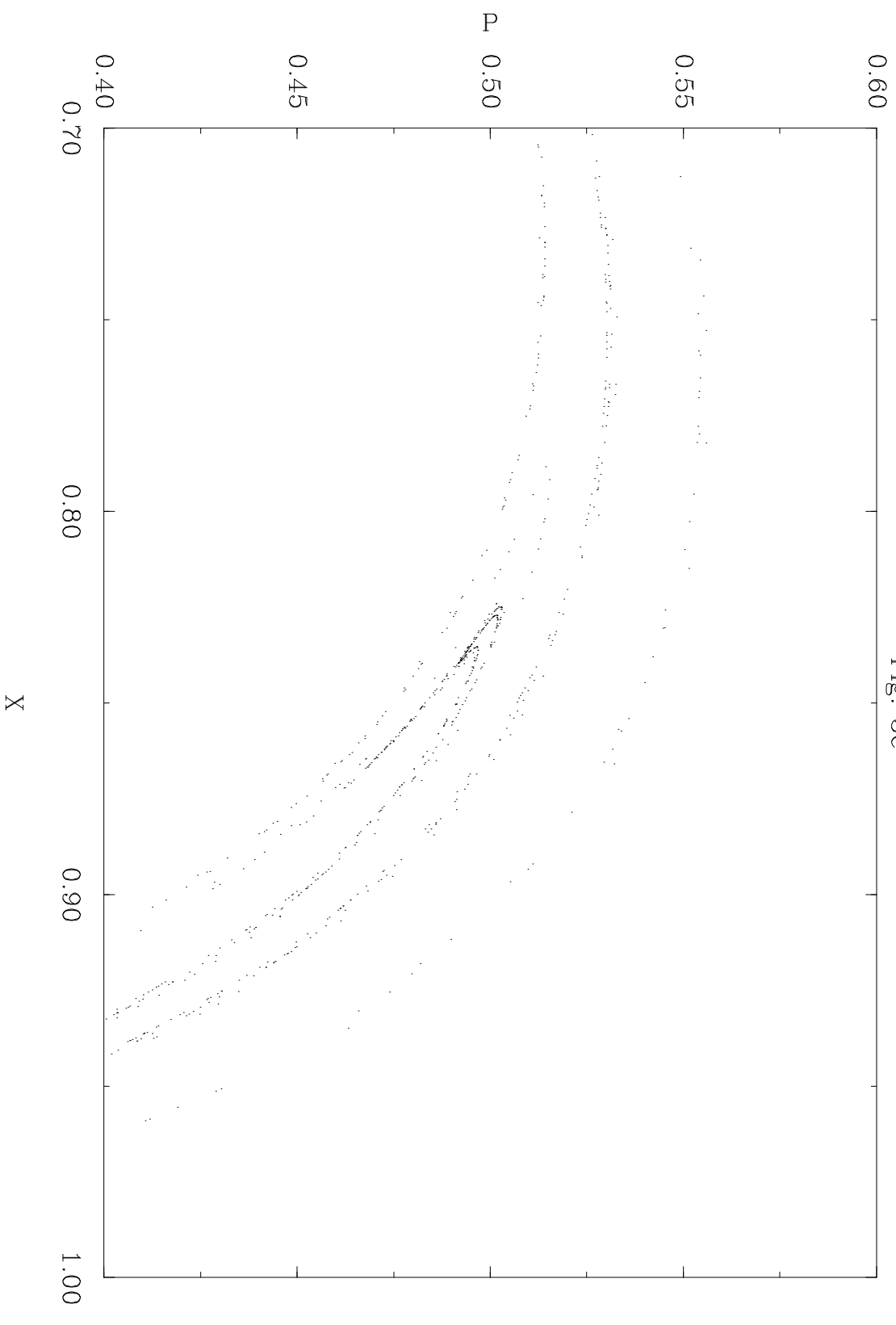
Poincare Section of Duffing Attractor

Fig. 3b



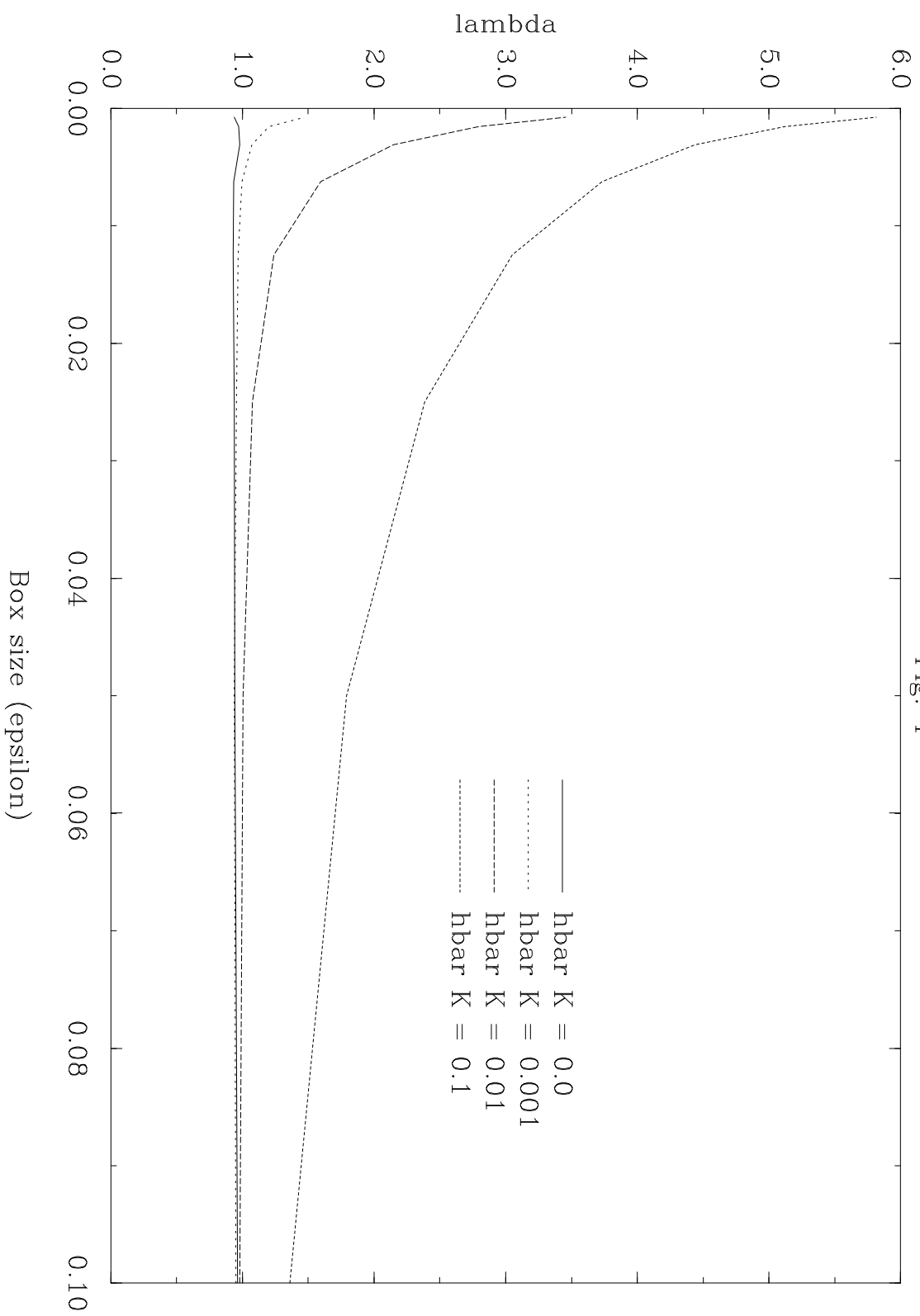
Poincare Section of Duffing Attractor

Fig. 3c



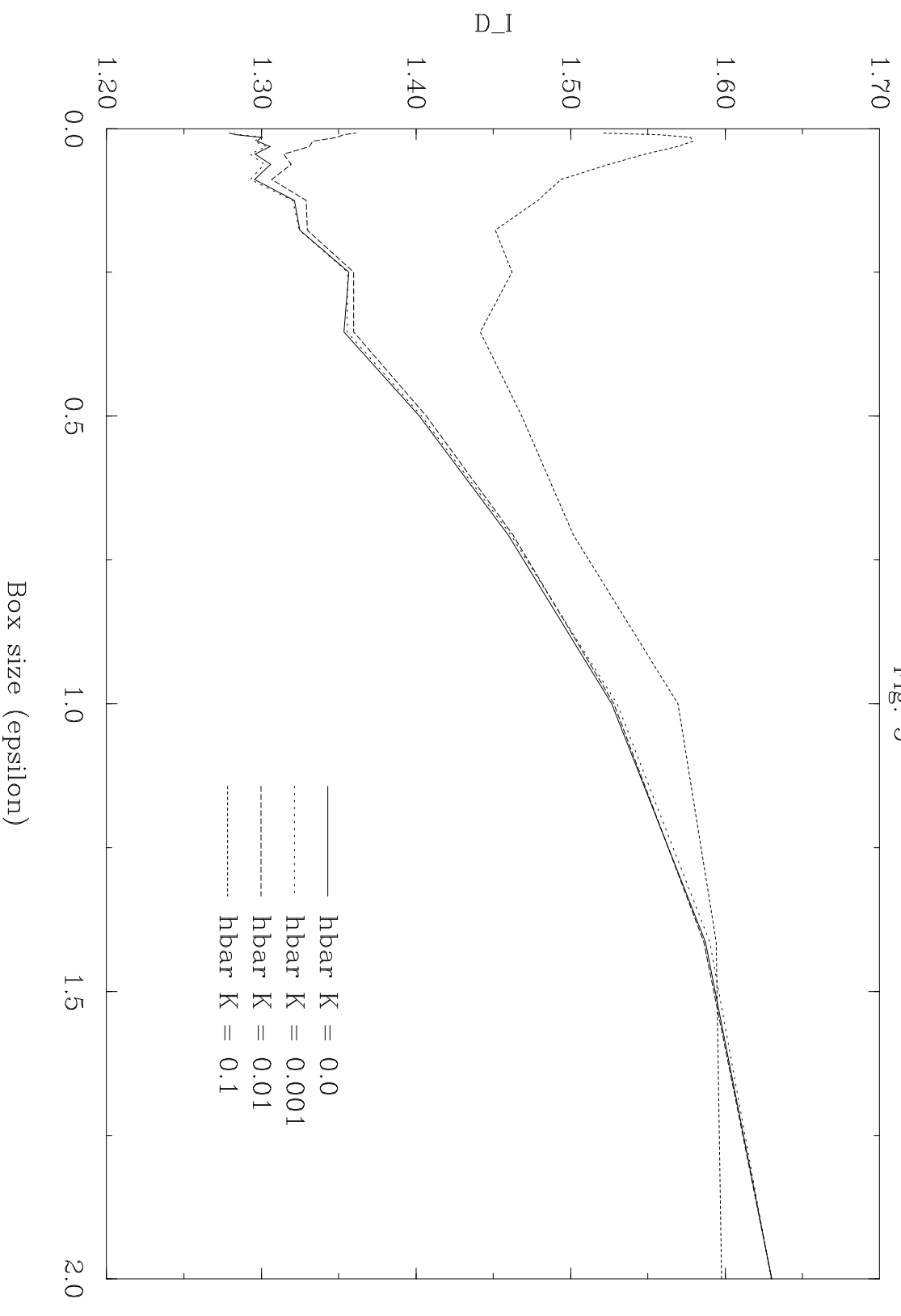
Lyapunov Exponent

Fig. 4



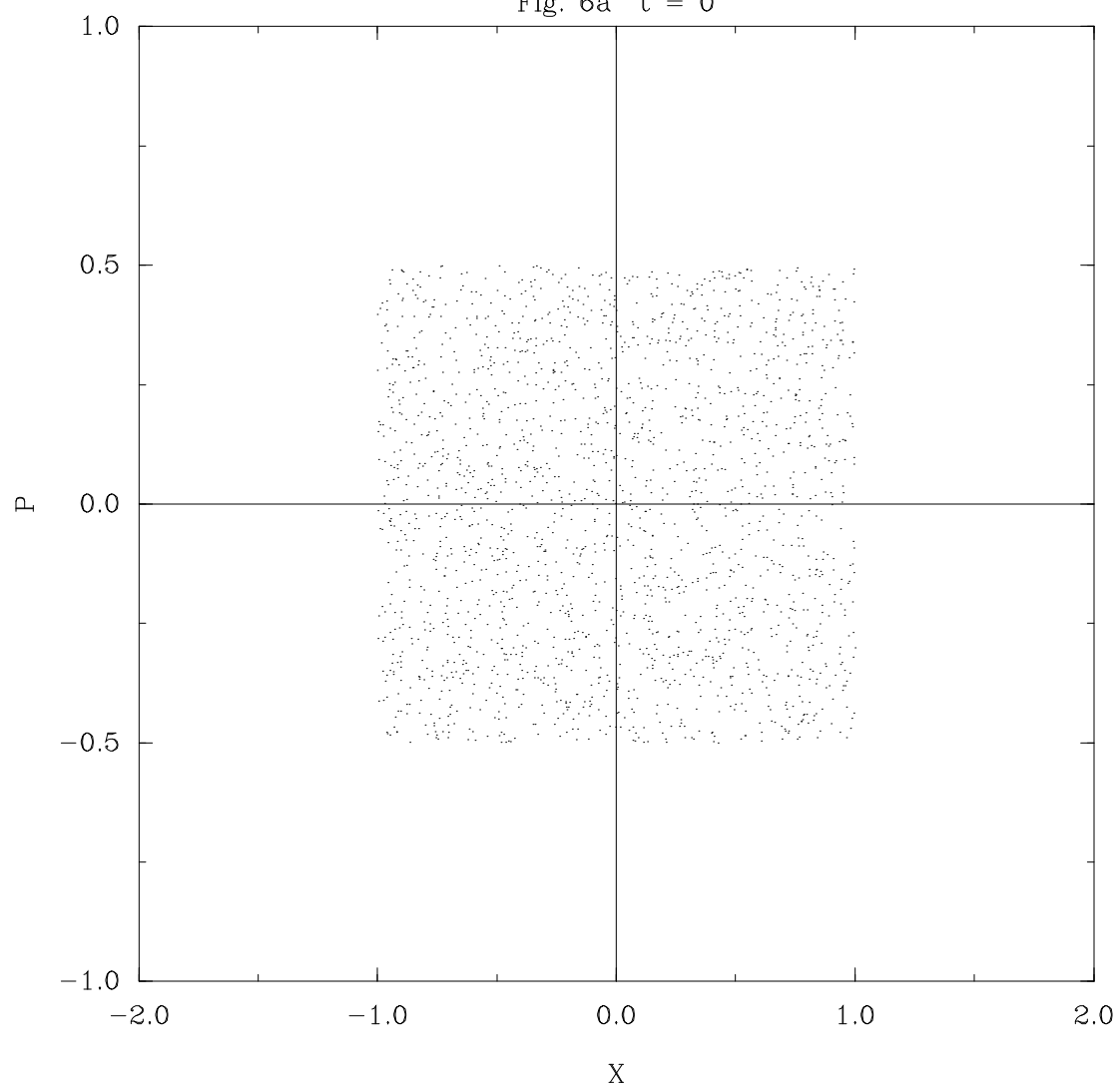
Information Dimension of attractor

Fig. 5



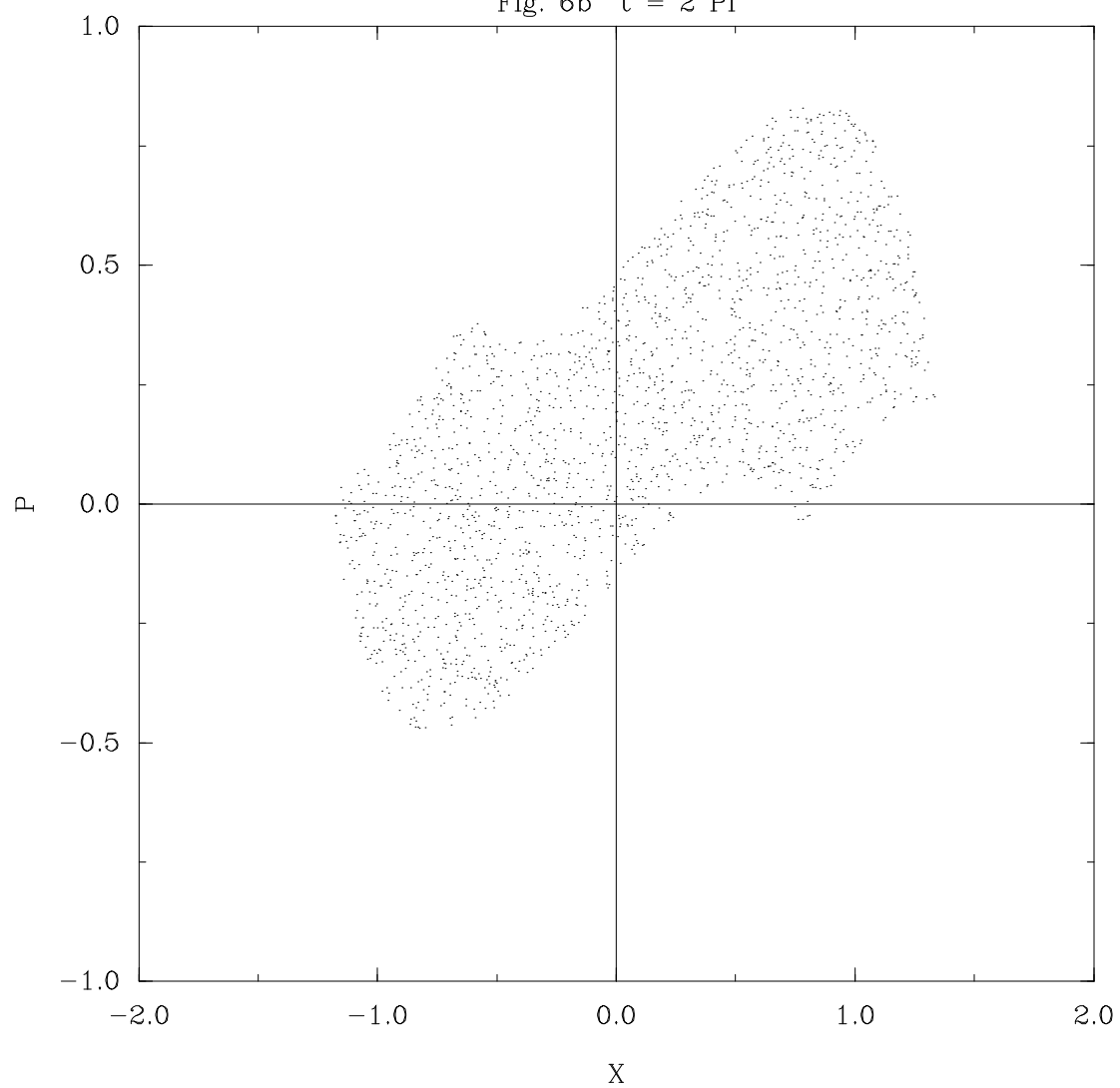
Evolution of Damped, Driven Duffing Oscillator

Fig. 6a $t = 0$



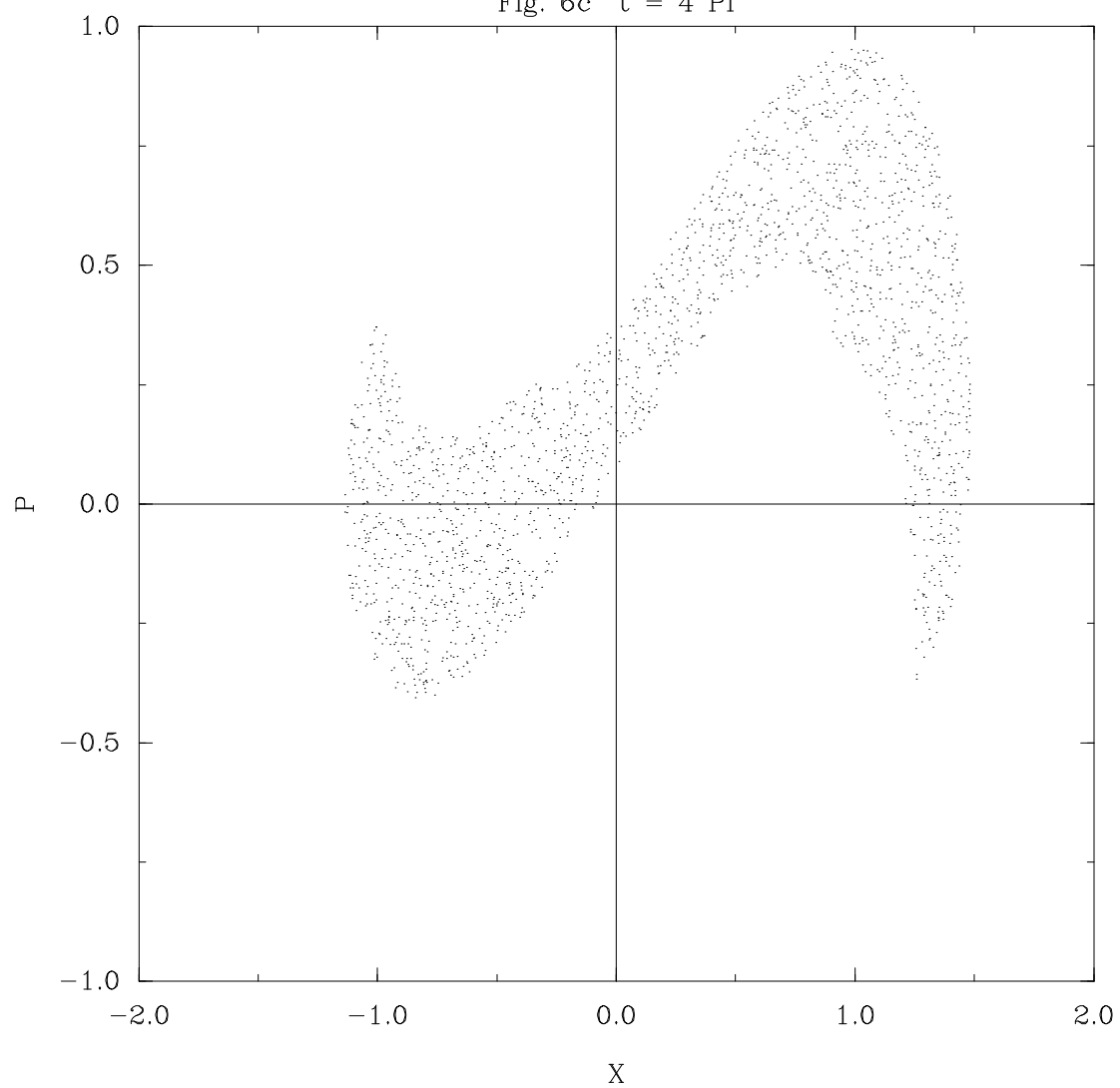
Evolution of Damped, Driven Duffing Oscillator

Fig. 6b $t = 2\pi$



Evolution of Damped, Driven Duffing Oscillator

Fig. 6c $t = 4 \text{ Pi}$



Evolution of Damped, Driven Duffing Oscillator

Fig. 6d $t = 20 \text{ Pi}$

